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Generation of planar quadrilateral mesh using tensegrity model

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Abstract

In this paper, we present a mechanical approach to shape generation of a roof surface discretized by planar quadrilateral mesh. The mesh is modeled by a tensegrity model consisting of the units with four cables and two struts. The units are guaranteed to be planar when the tensegrity model is in the self-equilibrium state. Hence, the generation of a planar quadrilateral mesh is transformed to the form-finding problem of the corresponding tensegrity model. The numerical examples demonstrate the high applicability of the proposed approach.

Keywords: architectural surface, planar quadrilateral mesh, tensegrity, form-finding.

1. Introduction

Contemporary architecture is increasingly embracing free-form shapes characterized by complex geometry. While creating digital models becomes easier with the highly advanced digital design tools nowadays, real-world fabrication and construction involve significant challenges (Pottmann [1]). Approximation of a free-form surface designed by architects using Planar Quadrilateral (PQ) meshes is a pivotal aspect of architectural design, especially for structures utilizing glass panels. Fabricating curved glass panels is notably costly, making the approximation of the surface to a PQ mesh particularly significant (see, e.g. Glymph et al. [2]). Although various approaches have been proposed for generating PQ meshes, most of them are very complex and require knowledge of computational geometry. Some traditional methods, for instance, the work by Liu et al. [3], involve initial computation of conjugate direction fields, followed by applying optimization techniques to achieve planarity.

This study overcomes difficulties in the conventional geometric approaches that are deeply rooted in discrete differential geometry. The main innovation of this study lies in our use of a tensegrity to model the mesh, ensuring the panels remain planar through its self-equilibrated configuration. This mechanical approach has not been previously studied, and it diverges significantly from the existing geometric approaches by employing the mechanical properties of planar tensegrity for this purpose. In the numerical examples, the proposed method is applied to form-finding of convex and non-convex roof surfaces.

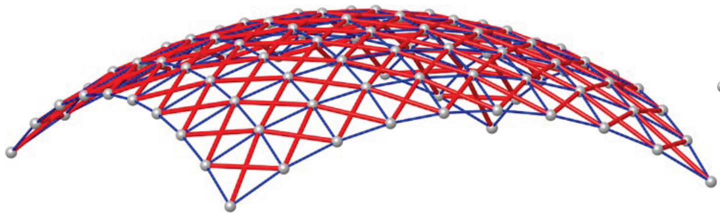
2. Methods

Suppose we are given a free-form surface that is discretized by a quad mesh; i.e., each panel has four straight edges and four nodes. The coordinates of the n nodes are assembled into the nodal coordinate vector $\mathbf{X}_0 \in \mathrm{R}^{3n}$. Our task is to find a new shape of the surface, in terms of the nodal coordinate vector $\mathbf{X}$ that is as close as possible to $\mathbf{X}_0$ on the one hand; and the four nodes of each panel are *co-planar* on the other hand.

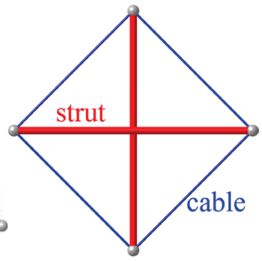
2.1. Tensegrity model

To adopt a mechanical approach in this study, the quad mesh is modeled by a tensegrity, for instance as shown in Figure 1(a) which will be revisited in Example 1. Each panel is modeled by a tensegrity unit as shown in Figure 1(b). The term tensegrity refers to the structure composed of continuous tensile members (cables) and discontinuous compressive members (struts) (Motro [4], Zhang and Ohsaki [5]).

The tensegrity unit consists of four cables outside and two diagonal struts inside. The cables in blue carry tension, and they correspond to the edges of the PQ panel. On the other hand, the struts in red carry compression, and they are the auxiliary members only for the purpose of guaranteeing the planarity of the unit. Note that the pair of struts are not connected with each other at their centers. The unit has one degree of statical indeterminacy; i.e., there exists one prestress mode that can self-equilibrate the structure, when no external load is applied. Because there are only three members; i.e., one strut and two cables, connected to each node, these members have to be coplanar in order to maintain its equilibrium state. It is notable that the prestresses in different units are independent of each other, except that every two neighboring units share the same cable. Therefore, the degree of statical indeterminacy is equal to the number of units of the tensegrity model of the surface. Hence, the generation of PQ mesh for a given architectural surface turns out to be the problem of finding the self-equilibrated configuration of its tensegrity model. This problem is called form-finding or shape-finding problem. In this study, we adopt the nonlinear analysis method which is suitable for form-finding of unstable structures with mechanisms [6].



(a) Tensegrity model



(b) Tensegrity unit

Figure 1: A PQ mesh modeled by the tensegrity structure in (a) with the quadrilateral unit in (b).

2.2. Form-finding

Suppose the tensegrity model has m members including cables and struts. Its equilibrium equation can be written as follows:

$$\mathbf{Ds} = \mathbf{f} \tag{1}$$

where $\mathbf{D} \in \mathrm{R}^{3n\times m}$ is the equilibrium matrix, $\mathbf{s} \in \mathrm{R}^{m}$ is the axial force vector, and $\mathbf{f} \in \mathrm{R}^{3n}$ is the out-of-balance force vector since we do not consider external loads. Note that boundary conditions are not given for constraining the rigid-body displacements and rotations. When $\mathbf{f}$ is a zero vector, the structure is in the self-equilibrium state, and all panels are planar according to the previous discussions.

The load vector $\mathbf{p} \in \mathrm{R}^{3n}$ is defined as follows, by incorporating the unbalanced force $\mathbf{f}$ and a small artificial load $\mathbf{p}_\mathrm{a} \in \mathrm{R}^{3n}$ directing from the current configuration $\mathbf{X}$ to the initial configuration $\mathbf{X}_0$ of the structure:

$$\begin{aligned} \mathbf{p} &= -\mathbf{f} + \mathbf{p}_\mathrm{a} \\ \mathbf{p}_\mathrm{a} &= \alpha(\mathbf{X}_0 - \mathbf{X}) \end{aligned} \tag{2}$$

where α is a small positive coefficient. The load vector $\mathbf{p}$ deforms the structure, and the configuration in terms of nodal coordinates $\mathbf{X}$ of the tensegrity model is updated as

$$\mathbf{X} \coloneqq \mathbf{X} + \mathbf{d} \tag{3}$$

where the displacement vector $\mathbf{d} \in \mathrm{R}^{3n}$ is estimated, as follows, by the genearlized inverse $\mathbf{K}^{-}$ of the tangent stiffness matrix $\mathbf{K} \in \mathrm{R}^{3n\times 3n}$ to find the minimum norm solution excluding the rigid body displacements and rotations:

$$\mathbf{d} = \mathbf{K}^{-}\mathbf{p} \tag{4}$$

2.3. Planarity error and geometry error

For the approximated tangent plane passing through the center $\mathbf{o}_i$ of the four nodes $\mathbf{x}_{i,j} \in \mathrm{R}^3$ $(j = 1,2,3,4)$ of panel i, its normal direction $\mathbf{n}_i \in \mathrm{R}^3$ is calculated by principal component analysis (Hoppe et al. [7]). The planarity of each panel is evaluated by the planarity error ε_i, which is defined as the quadric distance of its four nodes to the approximated tangent plane as

$$E_i = -4\left(\mathbf{o}_i^{\mathrm{T}}\mathbf{n}_i\right)^2 + \sum_{j=1}^{4}\mathbf{x}_{i,j}^{\mathrm{T}}\left(\mathbf{n}_i\mathbf{n}_i^{\mathrm{T}}\right)\mathbf{x}_{i,j} \tag{5}$$

The geometry error, which evaluates the error in approximating the target shape, is defined as follows

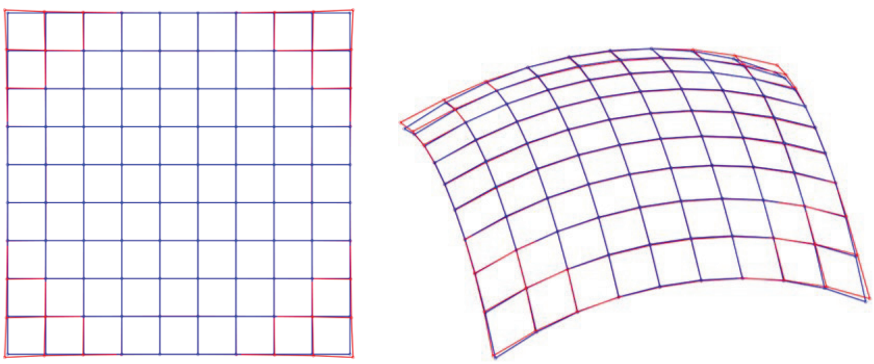
$$E_{\mathrm{g}} = ||\mathbf{X}_0 - \mathbf{X}||^2 \tag{6}$$

3. Numerical Examples

For all the examples, the axial stiffnesses of the struts and cables are 10^5 N/m and 10^3 N/m, respectively. The axial stiffnesses may have only limited influence on the final self-equilibrated configuration for PQ mesh. The initial forces of the struts are set to -10 N, and those of the cables are calculated by using the equilibrium matrix. The blue lines in the following figures refer to the edges of the initial shape, while the red ones are the self-equilibrated shape.

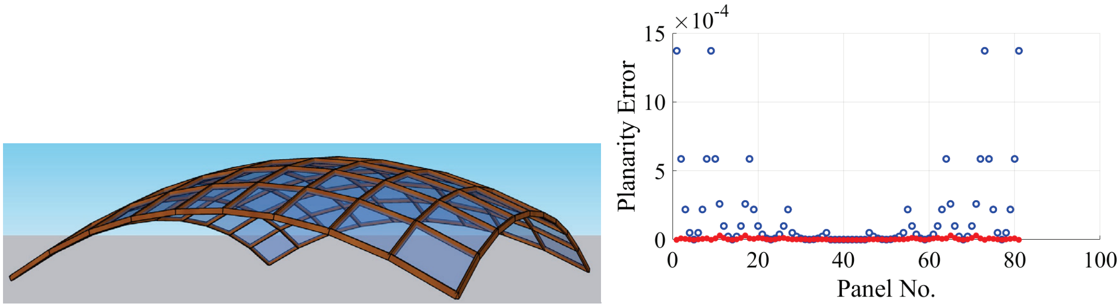
3.1. Example 1: Dome-shape surface

In Example 1, we consider a dome-shaped roof surface as shown in Figure 2, the tensegrity model of which is given in Figure 1(a). The spans of the initial shape are 18.0 m in both directions, and the height is 4.9 m. There are in total 100 nodes and 81 panels. For this example, the value of α in Equation (2) is set to zero. The final equilibrium shape with PQ mesh is shown in Figure 3(a). It can be observed in Figure 2 that the final shape is only slightly different from its initial one. Moreover, it is obvious from Figure 3(b) that the planarity error of each panel indicated by a blue circle at the initial shape is drastically reduced to a small positive value indicated by a red circle after equilibrium analysis for form-finding; i.e., all units become planar panels. The form-finding algorithm terminates within 59 steps, when the norm of $\mathbf{p}$ is less than 10^{-6} N.



(a) Top view (b) Side view

Figure 2: Dome-shape roof surface (Example 1, α=0.0). Blue lines: initial shape, red lines: final shape.



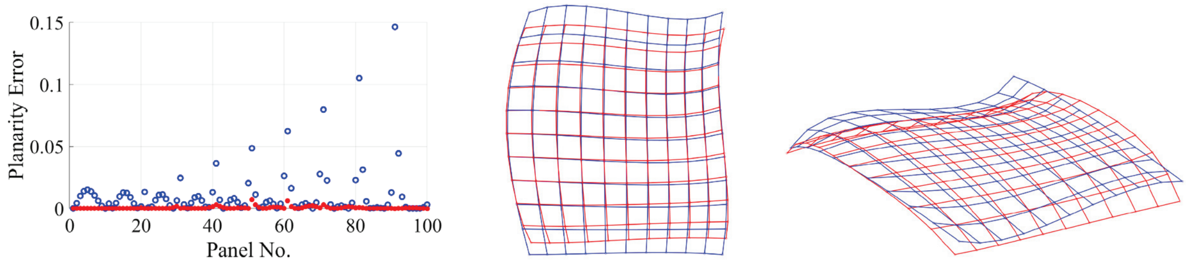
(a) Rendered view (b) Planarity error

Figure 3: Final shape and distribution of the planarity error (Example 1).

3.2. Example 2: Free-form surface

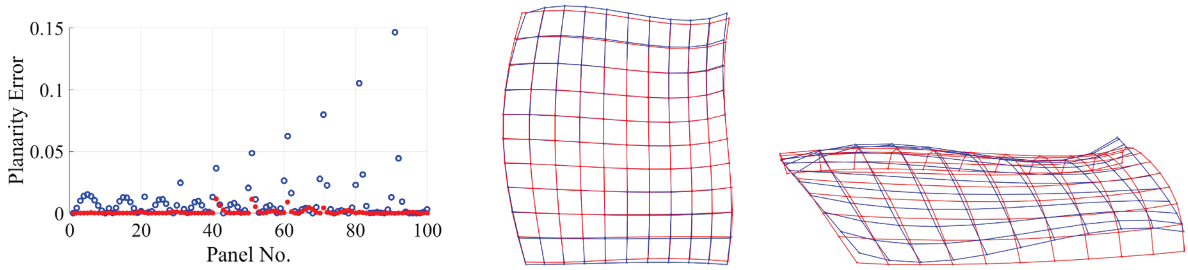
In Example 2, we consider a free-form roof surface generated by Bézier surface, as shown in Figure 4, as a target shape to be approximated by a PQ mesh. The span of the initial shape in one direction is 45.5 m, and that in the other direction is 50.1 m. The height of the structure is 7.9 m. There are in total 121 nodes and 100 panels. The form-finding algorithm terminates when the norm of $\mathbf{p}$ becomes less than 1.0 N.

Figures 4 and 5 respectively show the planarity error, top view, and side view of Example 2, with α=0.0 in Example 2(a) and $\alpha=10^{-3}$ in Example 2(b). The form-finding algorithm terminates within 409 steps in Example 2(a) and 177 steps in Example (b).



(a) Planarity error (b) Top view (c) Side view

Figure 4: Free-form surface of Example 2(a) (α=0.0). Blue lines: initial shape, red lines: final shape.



(a) Planarity error (b) Top view (c) Side view

Figure 5: Free-form surface of Example 2(b) ($\alpha=10^{-3}$). Blue lines: initial shape, red lines: final shape.

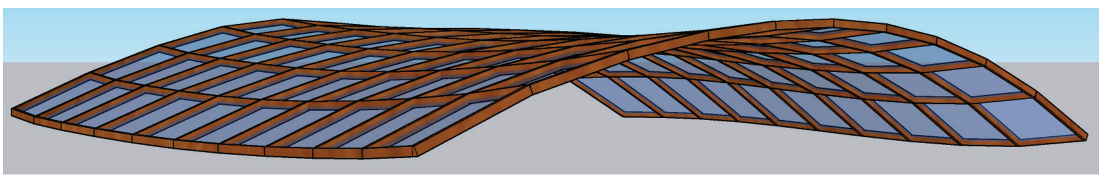


Figure 6: Final shape (Example 2(b)).

It can be observed from Figure 5 that the final shape adapts to have planar panels, with the planarity error close to zero at the final state indicated with red. The maximum planarity error is reduced from 0.1464 to 0.0071 in Example 2(a), and to 0.0117 in Example 2(b). In conclusion, some of the panels may have larger planarity errors, due to the guiding or inertia force $\mathbf{p}_\mathrm{a}$ in Equation 2 applied in addition to the out-of-balance force. On the other hand, the final configuration is closer to the specified target shape, reducing the geometry error from 1579.3 in Example 2(a) to 195.6 in Example 2(b). Hence, the guiding force could be adjusted by a trade-off between the shape approximation error and the planarity error. The rendered final shape of Example 2 is shown in Figure 6.

4. Conclusions

In this study, we introduced a mechanical method to address the challenge of approximating a roof surface, discretized by a PQ mesh. By ensuring planarity of each panel, we can significantly reduce the construction cost by eliminating the need for additional cost for panel fabrication. The mesh is modeled by a tensegrity model consisting of the units with four cables and two struts. It is guaranteed that the units are planar when the tensegrity model is in the self-equilibrium state. Hence, the generation of a planar quadrilateral mesh is transformed into the form-finding problem of the corresponding tensegrity model. Any well-developed form-finding methods for tensegrity can be utilized to solve the problem. The examples demonstrate the availability of the proposed approach to approximate various doubly curved surfaces. The same idea of using a planar tensegrity unit can be extended to the planar panels with different shapes.

Acknowledgments

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