Proceedings of the IASS 2024 Symposium
Redefining the Art of Structural Design
August 26-30, 2024, Zurich, Switzerland
Philippe Block, Giulia Boller, Catherine DeWolf,
Jacqueline Pauli, Walter Kaufmann (eds.)

Form finding of membrane shells via geometric stiffness methods: overview and characterization of well-posed problems

Andres F. Guerra Riaño*, István Sajtos*, Péter L Várkonyi*

* Department of Mechanics, Materials and Structures, Budapest University of Technology and Economics
Muegyetem rkp. 3, budapest, H-1111, Hungary
guerra.andres@edu.bme.hu

Abstract

Membrane theory assumes equilibrium through in-plane membrane forces, i.e. membrane shells resist applied loads without internal moments. Form-finding of membrane shell structures requires the solution of a partial differential equations (PDE) expressing static equilibrium in terms of stress and shape functions. We provide a systematic overview of existing approaches to the form-finding problem of membrane shells. This is followed by a detailed review of the Stress Density Method (SDM): a computationally efficient, and widely used method seeking shapes of equilibrium under prescribed 2nd Piola-Kirchoff stress field. Numerical algorithms typically consider the weak form of the equations or physical discretizations of the membrane surface, however we present the explicit, strong form of the governing equations. This approach allows us to find conditions of well-posedness, which guarantees uniqueness of the solution and continuous dependence on problem parameters. It is found that well-posedness depends on the character of the prescribed stress field. Some difficulties arising from attempts to solve ill-posed problems are illustrated.

Keywords: membrane structures, form finding, stress density method, partial differential equations, well-posedness, Pucher's equations.

1. Introduction

1.1. Structural from-finding

Membrane shells are thin surface structures, which balance external loads of arbitrary direction through in-plane membrane forces. They offer the possibility of visually attractive, and material-efficient structural solutions. The possibility of the membrane state of a shell depends on the geometry of its mid-surface, the type of load and its intensity, as well as on the thickness and the modulus of elasticity of the material i.e. the ability of deformation of the shell [1]. The curvature of the surface plays a critical role in equilibrium, hence the curvatures of membrane shells, and thus their global shapes need to be chosen carefully and in accordance with their loads.

Form-finding of membrane shells (also called initial equilibrium problem by some authors) is an essential part of the conceptual design of membrane shells, and it has a long history. Early approaches of R. Hooke, G. Poleni, and later A. Gaudi, H. Isler, and others used hanging models to find moment-free forms of structures. These models were made of cable elements or continuous surfaces (textile or soap film) representing arches, vaults, cable nets or membrane shells [2].

Computational form-finding methods can be viewed as virtual representations of physical models investigated with the aid of numerical solvers. Numerical methods take boundary conditions (support types and locations), topology, external loads, and some additional parameters (specific to the method used) as input. The solution of the problem of form-finding is a tuple of a structural shape and a stress field, which together ensure equilibrium of the structure under its load. Form-finding is followed by traditional steps of structural analysis. Many variants of the design process exist: for example, the load often depends on structural shape, in which case it becomes part of the solution; or form finding and structural analysis can be combined into an iterative process of optimal design.

1.2. Classification of computational methods

An important theoretical classification is based on the linear or nonlinear character of the equations to be solved. There are highly efficient and reliable computational algorithms to solve linear differential equations, or large systems of linear algebraic equations, and existence and uniqueness of the solution is often guaranteed [3]. Accordingly, linear form finding methods such as the Force Density and Stress Density Methods [4, 5] stand out for their low computational costs. In contrast, nonlinear problems needs refined and computationally expensive methods, and often fail to identify a solution.

Another well-known classification of computational form finding methods focuses on the solution strategy [2] with three important types. Structures with very low bending stiffness tend to deform under external loads into nearly funicular shapes. This observation allows one to use adapted versions of *nonlinear structural analysis* with large displacements using Finite Element Method for form finding [6, 7]. The outlined approach makes use of traditional tools available for structural analysis, but it is computationally often inefficient. In contrast, *geometric stiffness methods* attempt to find shapes consistent with funicular equilibrium without relying on material-dependent physically relevant stiffness parameters. Apparently, this class includes some highly efficient algorithms, which will be discussed in more detail in Section 2. Finally, *dynamic relaxation methods* solve the dynamic equation of a vibrating system, composed of the original model extended with actual or fictitious masses, and damping, see e.g. [8, 9]. Here the goal of the analysis is to identify a shape of stable equilibrium as the limit shape of damped dynamics. Dynamic relaxation methods can also be interpreted as iterative solution schemes of nonlinear structural analysis, or geometric stiffness methods.

Finally, a third important classification is based on distinction between continuous or discrete models. Discrete models include hinged bar systems, which is a natural model of cables, cable nets or vaults subject to concentrated loads, and the corresponding problems can be formulated as algebraic equations [10]. In contrast, form-finding of continuous models of membrane shells or funicular surface structures leads to partial differential equations, which must be discretized in order to be tackled by numerical solvers. Some forms of discretization are equivalent to a discrete physical hinged bar systems. For example equilibrium equations discretized using the finite difference method on an orthogonal plan grid, assuming zero shear stress has a possible physical interpretation as the equilibrium equations of a hinged bar system above plan orthogonal grid, loaded at the hinge points by a load equivalent to the load of the shell [11],[12]. In the literature of form-findig, continuous problems are often directly replaced by discrete physical models, which is reasonable from the point of view of numerical computation, However this transformation may hide important information about the problem, such as well-posedness, which is in the focus of the present work. Well-posedness is a basic concept of the theory of partial differential equations, which depends on the form of the equation as well as the types of boundary conditions. As we point out, well-posedness is crucial to the relevance of the solution found using computational methods to the original problem. However exact conditions of well-posedness are not available but in a few simple cases.

1.3. Structure of the paper

The rest of the paper is organized as follows: Section 2 is devoted to a detailed overview of geometric stiffness methods, among which the stress density method (SDM) stands out for its linearity and computational efficiency. In Section 3, we present the barely used *strong form* of the SDM, which allows us to investigate conditions of well-posedness and to highlight the importance of this property in structural form-finding (Section 4). The paper is closed by a discussion of open questions (Section 5).

2. An overview of geometric stiffness methods for membrane shells

Geometric stiffness methods aim to find a shape satisfying conditions of static equilibrium. The statics approach means that deformations and stiffness of the structure are not taken into consideration. In most cases, boundary types, structural topology and loads (or some relation between shape and loads) are prescribed, and a large set of possible solutions exists. Different form-finding methods specify different additional parameters in order to arrive to a unique solution. Such parameters may include various features of the shape or the stress field in the structure. In what follows we outline three families of methods along with their main strengths.

2.1. Form-finding under prescribed plan view

In architectural design, a common strategy of preliminary design is to prescribe orthogonal projection of the shell to a horizontal plane (i.e. plan view) along with the arrangement of supports.

Some of the earliest geometric stiffness methods [11] prescribed in a first step the projection of the shape into horizontal plane as well as the projection of membrane stresses or bar forces into the same plane. Continuous problems were discretized. Then, the equilibrium of horizontal force components for a prescribed projected bar configuration was investigated. For free edges, the projected bar configuration geometry often needed modification to satisfy the static boundary conditions. The height of the structure was found in a second step by numerically solving the equations of equilibrium (which is a linear problem). These works focused on a limited class of cases, where a matching triplet of plan view form, boundary conditions, and projected stress distribution could be found analytically. For discrete models, this approach was later reformulated and became known as Thrust Network Analysis (TNA) [13].

More recent work by Chiang et. al. [14] on continuous problems extends these methods in order to deal with situations where a matching triplet of projected shape, stress, and boundary conditions cannot be found in closed form. Chiang's method prescribes plan view and boundary conditions. Starting from an initial guess of a matching stress distribution, an intricate iteration process is used to solve the nonlinear PDE, whose solution is the corresponding projected stress distribution. Then the elevation, and 3D stress state are found as before. An important weakness of this approach is the possible divergence of the iteration process, which stems from its nonlinearity.

2.2. Form-finding under prescribed Cauchy stress

The structural designer often aims to control the Cauchy stress distribution inside the shell in order to construct a structural solution with optimized strength everywhere. Under prescribed Cauchy stress, the equilibrium equations become nonlinear PDEs. Iterative solution procedures were proposed by [15][16] in order to achieve generate minimal surfaces characterized by uniform Cauchy stress with prescribed boundary conditions. A well-known weakness of this method is the non-existence of solutions for some boundary conditions.

2.3. Form-finding with low computational cost

The Force Density Method (FDM) [4, 17] was initially developed for discrete cable networks. The method seeks a geometry providing equilibrium under prescribed ratio of normal force to length in each bar. The equilibrium equations are linear in the unknown joint coordinates, which allows computationally efficient solution without iteration. In the case of free boundary, the method automatically provides the corresponding edge geometry. In turn, the method does not allow direct control over plan view form or Cauchy stresses.

The Stress Density Method (SDM) is an analogous procedure for continuous models (preserving linearity) first proposed by Haber and Abel [18]. This method takes as input a triplet of a *reference configuration*, boundary conditions, and a proposed 2nd Piola-Kirchoff (2PK) stress field over the reference configuration. The solution of the problem is a shape of equilibrium such that constraints are obeyed, and 2PK stresses match the prescribed distributions. The weak (variational) form of the PDE underlying the SDM was derived and solved using Finite Element Methods by [18, 15]. Other works [19, 20, 21] focus on physical discretizations of the continuous membrane surface and the corresponding algebraic equations. It appears to us that the strong (explicit) PDE form of the SDM equations has not been presented and investigated in the literature. In the current paper, we will derive the corresponding PDE from equilibrium equations of an infinitesimal surface element, which enables us to study possible ill-posedness of the problem and its practial consequences.

3. Explicit form of the stress density method

A model of the SDM (Fig. 1) has four crucial prescribed components:

- a bounded domain $\mathbb{D}$ of the plane parametrized by two scalars (p, s) (reference configuration)
- a set of kinematic constraints (supports) and static boundary conditions (e.g. specification of free edges without external support).
- a vector-valued external load $g(p, s)$ over $\mathbb{D}$ (which is by default a force distributed on a surface but it may include linearly distributed or concentrated forces as well).
- three scalar functions $\xi(p, s)$, $\tau(p, s)$, $\eta(p, s)$ over $\mathbb{D}$ representing components of the prescribed 2PK stress resultant tensor with respect to the local coordinate system of the reference configuration:

$$Q(p, s) = \begin{bmatrix} \xi(p, s) & \tau(p, s) \\ \tau(p, s) & \eta(p, s) \end{bmatrix}$$

The aim of the SDM is to find an unknown shape function $r(p, s) : \mathbb{D} \to \mathbb{R}^3$, as well as three scalar functions representing the (Cauchy) stress resultants.

Let (x, y, z), and (g_x, g_y, g_z) denote components of r and g, respectively. By definition of 2PK stress, the stress resultants acting on an infinitesimal surface element in the equilibrium configuration can be obtained by transforming the force vectors corresponding to ξ, η, τ as $\rho \to J_r \rho$ where matrix $J_r = [r_p \; r_s]$ is the Jacobian of the form function. The free body diagram of the surface element is depicted in

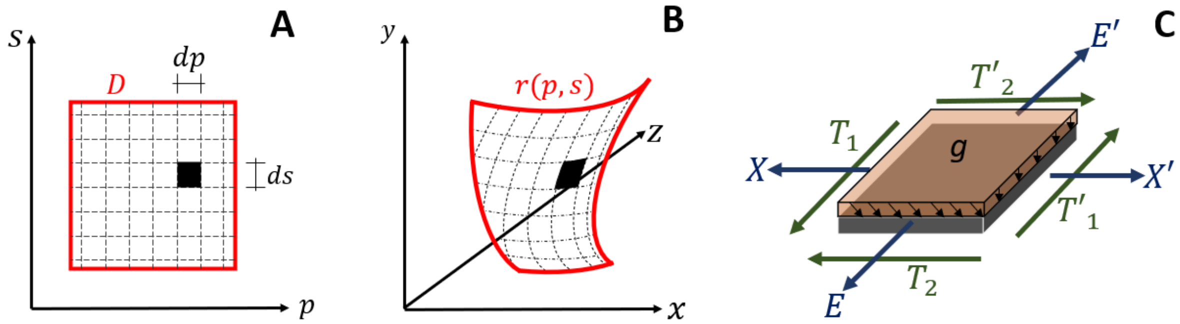

Figure 1: SDM model of a membrane shell with reference configuration (A), equilibrium shape (B), and free body diagram of an infinitesimal piececorresponding to the black quadrangle in panel B (C). The arrows in panel C represent resultants of forces distributed along the edges of the surface element.

Fig. 1C, and uses the following notation

$$X = \xi(p,s)r_p(p,s)ds, X' = \xi(p+dp,s)r_p(p+dp,s)ds \quad (1)$$

$$E = \eta(p,s)r_s(p,s)dp, E' = \eta(p,s+ds)r_s(p,s+ds)dp \quad (2)$$

$$T_1 = \tau(p,s)r_s(p,s)ds, T_1' = \tau(p+dp,s)r_s(p+dp,s)ds \quad (3)$$

$$T_2 = \tau(p,s)r_p(p,s)dp, T_2' = \tau(p,s+ds)r_p(p,s+ds)dp \quad (4)$$

Note that the shape of the surface element is a paralellogram embedded in 3D space, and the forces X, E, T_1, T_2 are parallel to its edges. The equilibrium of the surface element is expressed as

$$\frac{\partial}{\partial p}[\xi\frac{\partial r}{\partial p}] + \frac{\partial}{\partial s}[\eta\frac{\partial r}{\partial s}] + \frac{\partial}{\partial s}[\tau\frac{\partial r}{\partial p}] + \frac{\partial}{\partial p}[\tau\frac{\partial r}{\partial s}] + g = 0 \quad (5)$$

We use the product rule, and introduce a shorthand notation for partial derivatives in the form of lower indices to obtain:

$$\xi_p r_p + \xi r_{pp} + \eta_s r_s + \eta r_{ss} + \tau_p r_s + \tau_s r_p + 2\tau r_{ps} + g = 0 \quad (6)$$

This equation is linear, furthermore, it can be decomposed to 3 independent PDEs for (x, y, z). The three equations are identical except for the differences in load components and boundary conditions. For example, the $z(p, s)$ coordinate of the surface is the solution of

$$\xi_p z_p + \xi z_{pp} + \eta_s z_s + \eta z_{ss} + \tau_p z_s + \tau_s z_p + 2\tau z_{ps} + g_z = 0. \quad (7)$$

4. Well-posedness of PDE problems and form-finding

The explicit form given in Equation 7 helps us to apply the theory of PDEs to this problem. Linear, second order partial differential equations can be classified according to their discriminants, which allows to determine their transformation into a canonical form at any given point (p_0, s_0) [22]. For (7), the discriminant is

$$L = \tau(p_0, s_0)^2 - \xi(p_0, s_0)\eta(p_0, s_0), \quad (8)$$

which depends on the prescribed 2PK stress tensor. If L is equal to zero, Eq. (7) is parabolic; if $L < 0$ then it is elliptic and if $L > 0$ then it is hyperbolic. At this point, it is worth noting that elliptic equations correspond to prescribed principal Cauchy stresses with identical signs (i.e. shells under pure compression or pure tension), whereas hyperbolic equations correspond to prescribed principal stresses with opposite sign (shell subject to compression in one direction but tension in the orthogonal direction).

ξ, η, and τ are often prescribed as constant values, meaning that the problem keeps the same canonical classification across the entire domain. In the elliptic case, constant stresses transform (7) to Laplace's equation for unloaded shells ($g_z = 0$) or Poisson's equation for the loaded case ($g_z \neq 0$). In the hyperbolic case, Eq. 7 with constant coefficients is the standard Wave equation.

The boundary conditions of (7) depend on how the boundaries are supported. Fixed supports at the boundaries correspond to prescribed position coordinates, i.e. standard Dirichlet boundary condition in the nomenclature of PDE theory. Free membrane edges imply prescribed (vanishing) forces across the boundaries, which can be expressed for an edge with normal vector n in the reference configuration as $J_r Q n = 0$, where J_r is the Jacobian of the shape function r. This boundary condition is closely related to the standard Neumann boundary condition $J_r n = 0$ of PDE theory, i.e. a prescribed directional derivative of the unknown function normal to the boundary. For some values of Q and n (including the two examples presented below), the boundary condition of a free boundary is identical to Neumann boundary condition.

Together, the PDE and its boundary conditions determine a crucial property of the problem: its well-posedness. Well-posedness means [23] that:

- *Existence and uniqueness*: there is exactly one solution $z(p, s)$ which satisfies the PDE and all boundary conditions.
- *Stability*: the solution undergoes small changes in response to small changes of the PDE or its boundary conditions .

Some conditions of well-posedness are known for linear PDEs of order 2 with standrd types of boundary conditions. A boundary value problem (BVP) with one boundary condition (either Dirichlet or Neumann) along all boundaries of a closed domain is well-posed if the equation is elliptic in all of its points [24, 25], and ill-posed for a uniformly hyperbolic case [24, 26]. These mathematical results are applicable to structural problems, in which every point along the boundary of $\mathbb{D}$ is designated as free or fixed, with Dirichlet or Neumann boundary condition at all of those points.

As a simple example of a well-posed problem, consider a shell with a unit square $0 \leq p \leq 1$ and $0 \leq s \leq 1$ as reference configuration. Assume that all points of the square's edges are fixed at $(x, y, z) = (p, s, 0)$, i.e. in this case the reference configuration is identical in shape and size to the prescribed arrangement of the supports. The shell is subject to a constant distributed load $g(p, s) = (0, -2, -5)$ applied over the whole shell area. Besides, constant 2PK stress $\{\xi, \eta, \tau\} = \{-1, -1, 0\}$ is prescribed, thus the discriminant L of eq. (7) and of the analogous equations for x and y are negative. These elliptic BVPs can be rewritten as:

$$x_{pp} + x_{ss} = 0;\; x(0, s) = 0, x(1, s) = 1, x(p, 0) = p, x(p, 1) = p \tag{9}$$

$$y_{pp} + y_{ss} - 2 = 0;\; y(0, s) = s, y(1, s) = s, y(p, 0) = 0, y(p, 1) = 1 \tag{10}$$

$$z_{pp} + z_{ss} - 5 = 0;\; z(0, s) = 0, z(1, s) = 0, z(p, 0) = 0, z(p, 1) = 0 \tag{11}$$

We generated the solutions of all 3 boundary value problems by using a numerical BVP solver *solvepde* using Finite Element Method, available through the Partial Differential Equation Toolbox of the MatLab software package. The shell geometry encoded by the solutions is shown in Figure 2A. A modified version of the problem in which two boundary edges parallel to the p axis are prescribed to be free can be formulated as

$$x_{pp} + x_{ss} = 0;\ x(p,0) = p, x(p,1) = p;\ x_s(0,s) = x_s(1,s) = 0 \tag{12}$$

$$y_{pp} + y_{ss} - 2 = 0;\ y(p,0) = 0, z(p,1) = 1;\ y_s(0,s) = y_s(1,s) = 0 \tag{13}$$

$$z_{pp} + z_{ss} - 5 = 0;\ z(p,0) = 0, z(p,1) = 0;\ z_s(0,s) = 0; z_s(1,s) = 0. \tag{14}$$

These equations are again well-posed, elliptic BVPs. The resulting geometry is shown in Figure 2B. As a relatively fine mesh was used by the solver, both solutions are reasonable approximations of the smooth solution of the original PDEs. Hence, the form-finding problem has been solved successfully.

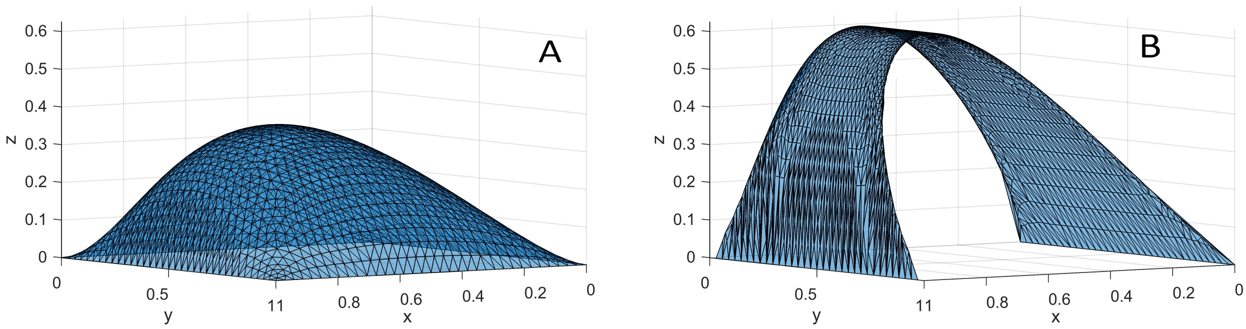

Figure 2: Examples of numerically obtained solutions for elliptic 2PK stress fields with Dirichlet boundary conditions (A) and mixed Dirichlet and Neumann boundary conditions (B) both yielding well-posed problems.

Consider now modified versions of the previous examples in which the prescribed stress values are $\{\xi, \eta, \tau\} = [1, -1, 0]$. Then the discriminant (8) is positive,i.e. (7) and the analogous equations for x and y are hyperbolic. Hence this problem is ill-posed. Numerical solutions generated by the same numerical BVP solver are illustrated by Figure 3. What we see now is irregular geometry, which extends far beyond the supports, and is not a viable candidate for the shape of a membrane shell. This example illustrates an important attribute of ill-posed problems: extreme sensitivity to small changes in the equations. As discretization schemes can be viewed as perturbations of the continuous problem to be solved, the solution of the discretized problem cannot be considered a reasonable approximation of the exact solution. Interestingly, the example that was presented here is a simple wave equation, which probably has no solution or non-unique solution depending on the prescribed loads [27].

It is important to emphasize that form finding with other types of boundary conditions is also possible, and may result in different conditions of well-posedness. For example, Cauchy boundary condition means that 0,1, or 2 boundary conditions are prescribed at appropriately chosen segments of the boundary (details omitted), which means that we have some boundaries with simultaneous conditions for positions and reaction forces, but no specification at all at other segments of the boundary. This kind of boundary condition makes uniformly hyperbolic problems well-posed, but elliptic ones ill-posed [28].

5. Discussion

Within the wide range of computational form-finding methods, material-independent Geometric Stiffness methods are especially popular. Inside this category, one can distinguish between numerous methods, which address various needs of designers. Taking into account that the structural form and its internal stresses are strongly related, the designer can choose two basic alternatives to find shapes in static equilibrium: one choice is to have direct control over the shape (e.g. plan view) but not over the

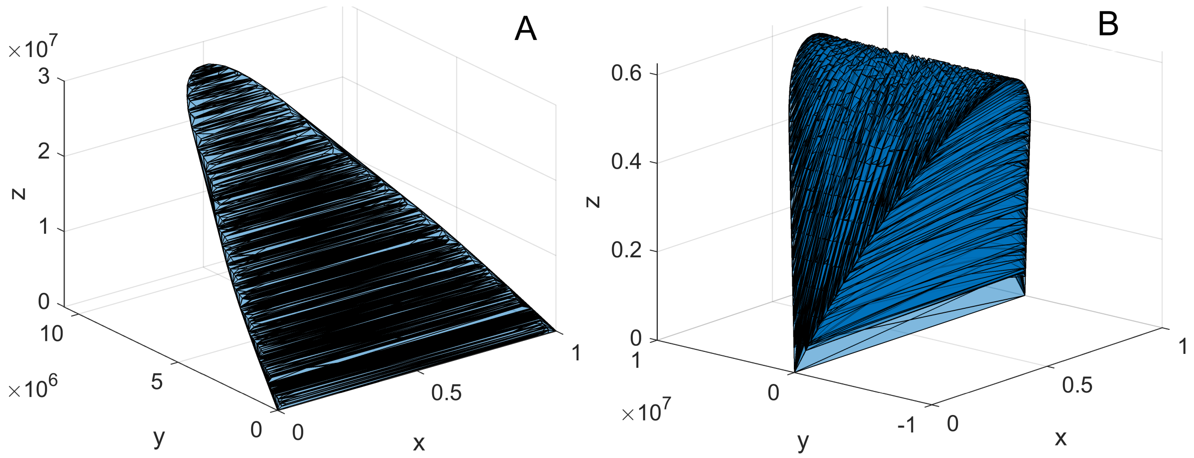

Figure 3: Examples of numerically obtained solutions for hyperbolic 2PK stress fields with Dirichlet boundary conditions (A) and mixed Dirichlet and Neumann boundary conditions (B), both yielding ill-posed problems.

Cauchy stress tensor. Alternatively, one can choose to prescribe external loads and a Cauchy stress along with boundary conditions at the price of loosing direct control over shape.

Nevertheless, there is an intermediate approach, which is important due to its low computational cost. The key idea of these methods is to generate linear equations, as linear approaches stand out for their computational efficiency compared to non-linear methods. In this case the shape is found under a prescribed boundary conditions, external loads and the ratio of normal force to length for discrete problems (Force Density Method), or a 2PK stress tensor over a reference configuration (Stress Density method). There is no direct control over the shape or the Cauchy stress.

As the Force Density Method uses sets of linear equilibrium equations, the funicular shape can be found using simple algebraic operations. Similarly, for the Stress Density Method, discretization required by numerical solvers yields linear algebraic equations. However studying the original continuous problem reveals important aspects of this numerical solution, in particular the well-posedness of the problem, which directly impacts the relevance of the discrete solution. This work focused on the Stress Density method, for which some conditions of well-posedness are available in the mathematical literature. Nevertheless, other form finding methods face similar limitations as the continuous problem is formulated using partial differential equations.

Well-posedness depends on the PDE as well as on the boundary conditions. Regarding the Stress Density method, conditions of well-posedness are established for completely elliptic or completely hyperbolic 2PK stress fields and special types of boundary conditions. However, in other cases, the lack of an underlying mathematical theory makes challenging to study the significance of discrete solutions.

Acknowledgments

This work has been supported by the National Research, Development and Innovation Fund of the Ministry of Culture and Innovation of Hungary under project K143175.

References

[1] C. Truesdell, “The membrane theory of shells of revolution,” *Transactions of the American Mathematical Society*, vol. 58, pp. 96–166, 1945.

[2] S. Adriaenssens, P. Block, D. Veenendaal, and C. Williams, *Shell structures for architecture: form finding and optimization*. Routledge, 2014.

[3] T. L. Saaty, *Modern nonlinear equations*. Courier Corporation, 2012.

[4] H.-J. Schek, "The force density method for form finding and computation of general networks," *Computer methods in applied mechanics and engineering*, vol. 3, no. 1, pp. 115–134, 1974.

[5] A. F. Guerra Riaño and P. L. Várkonyi, "Form-finding using the force density method: Existence of solutions, singularities, and an analogy to electric circuits," *International Journal of Space Structures*, vol. 38, no. 4, pp. 302–326, 2023.

[6] M. Pagitz and J. M. Tur, "Finite element based form-finding algorithm for tensegrity structures," *International Journal of Solids and Structures*, vol. 46, no. 17, pp. 3235–3240, 2009.

[7] J. Lienhard, "Bending-active structures: Form-finding strategies using elastic deformation in static and kinetic systems and the structural potentials therein," 2014.

[8] A. Kilian and J. Ochsendorf, "Particle-spring systems for structural form finding," *Journal of the international association for shell and spatial structures*, vol. 46, no. 2, pp. 77–84, 2005.

[9] D. Hegyi, I. Sajtos, G. Geiszter, and K. Hincz, "Eight-node quadrilateral double-curved surface element for membrane analysis," *Computers and Structures*, vol. 84, no. 31–32, pp. 2151–2158, 2006.

[10] J. Szabó and L. Kollár, *Structural design of cable-suspended roofs*. Akadémiai Kiadó and Ellis Horwood, 1984.

[11] J. Pelikan, "Membrane structures," in *Proceedings of the Second Symposium on Concrete Shell Roof Construction, Oslo*, Teknisk Ubekland, Oslo, 1958, pp. 229–231.

[12] J. Pelikan, "Form-determination of braced domes," in *ed. R.M. Davies: Space Structures: The International Conference on Space Structures, Surrey*, Blakwell Scientific Publication, Oxford, 1967, pp. 160–164.

[13] P. Block and J. Ochsendorf, "Thrust network analysis: A new methodology for three-dimensional equilibrium," *Journal of the International Association for shell and spatial structures*, vol. 48, no. 3, pp. 167–173, 2007.

[14] Y.-C. Chiang and A. Borgart, "A form-finding method for membrane shells with radial basis functions," *Engineering Structures*, vol. 251, p. 113 514, 2022.

[15] K.-U. Bletzinger and E. Ramm, "A general finite element approach to the form finding of tensile structures by the updated reference strategy," *International Journal of Space Structures*, vol. 14, no. 2, pp. 131–145, 1999.

[16] K.-U. Bletzinger, R. Wüchner, F. Daoud, and N. Camprubí, "Computational methods for form finding and optimization of shells and membranes," *Computer methods in applied mechanics and engineering*, vol. 194, no. 30-33, pp. 3438–3452, 2005.

[17] K. Linkwitz and H. Sheck, "The force density method for formfinding and computation of networks," *Computer Methods in applied Mechanics and Engineering*, vol. 3, pp. 115–134, 1974.

[18] R. Haber and J. Abel, "Initial equilibrium solution methods for cable reinforced membranes part i—formulations," *Computer Methods in Applied Mechanics and Engineering*, vol. 30, no. 3, pp. 263–284, 1982.

[19] B. Maurin and R. Motro, "The surface stress density method as a form-finding tool for tensile membranes," *Engineering structures*, vol. 20, no. 8, pp. 712–719, 1998.

[20] R. M. Pauletti and P. M. Pimenta, “The natural force density method for the shape finding of taut structures,” *Computer Methods in Applied Mechanics and Engineering*, vol. 197, no. 49-50, pp. 4419–4428, 2008.

[21] R. M. O. Pauletti and F. L. Fernandes, “An outline of the natural force density method and its extension to quadrilateral elements,” *International Journal of Solids and Structures*, vol. 185, pp. 423–438, 2020.

[22] Y. Pinchover and J. Rubinstein, *An introduction to partial differential equations*. Cambridge university press, 2005, vol. 10.

[23] W. A. Strauss, *Partial differential equations: An introduction*. John Wiley & Sons, 2007.

[24] J. Hadamard, “Sur les problèmes aux dérivées partielles et leur signification physique,” *Princeton university bulletin*, pp. 49–52, 1902.

[25] B. E. Kanguzhin and A. Aniyarov, “Well-posed problems for the laplace operator in a punctured disk,” *Mathematical Notes*, vol. 89, pp. 819–829, 2011.

[26] D. W. Fox and C. Pucci, “The Dirichlet problem for the wave equation,” *Annali di Matematica pura ed applicata*, vol. 46, no. 1, pp. 155–182, 1958.

[27] D. Bourgin and R. Duffin, “The Dirichlet problem for the vibrating string equation,” *Bulletin of the American Mathematical Society*, vol. 45, no. 12, pp. 851–858, 1939.

[28] A. N. Tikhonov and V. Arsenin, *Solutions of ill-posed problems*. V. H. Winston and Sons, 1977.