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On graphical methods applied to cantilevering forms

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Abstract

This paper will make use of graphic statics, derived from 19th century and earlier techniques, to analyse and form-find cantilevering gridshell structures. The proposed method discretises continuous shell surfaces into triangulated mesh inputs, and demonstrates the applicability of "2.5D" graphic statics combined with the Force Density Method (FDM) on these mesh inputs, to determine the Airy stress function for loaded cantilevering surfaces for a given set of vertical loads. Similarly, Airy stress functions may be used to form-find loaded cantilevering gridshells. The paper will investigate the effects of manipulating the Airy stress function for a given shell structure and vertical loading, on the form-found results, and will outline observed limitations of the process. Structures and results derived using graphical methods will be verified using more traditional finite-element analysis methods. Cantilevering structures constitute an interesting structural typology that has been given limited consideration to-date, in research into graphical methods. Analysing these types of structures may require dealing with hyperbolic differential equations with associated difficulty and complexity. The methods described in the paper are relatively simple, direct, and flexible, and do not make use of iterative computation to arrive at solutions.

Keywords: graphic statics, form-finding, Maxwell, cantilevering structures, computational design, gridshell design

1. Introduction

Graphic statics refers to the 19th century technique for the analysis of load-bearing 2D trussed structures. The graphical methods emerged in response to the development of cast-iron and early steel bridges [1], and were developed by Culmann, Maxwell, Rankine and others based on the earlier work by mathematicians, physicists, and polymaths such as Hooke [2] and Varignon [3]. The theory underpinning graphic statics is founded on the reciprocal nature of form and force observed by Maxwell [4]. Geometric constructions may be used to determine forces for a corresponding form, or similarly, allow for the form to be determined for a set of corresponding forces.

Structural trusses are 2D projections of 3D plane-faced polyhedra, which are continuous piecewise-linear Airy stress functions. The change in slope between faces at a shared boundary of the Airy stress function defines the axial force in the corresponding element, while the regions between faces are planar, representative of the absence of force between elements. The 2D truss structure is thus the projection of a range of possible 3D polyhedral Airy stress functions, each of which leads to a distinct force diagram. The range of possible Airy stress functions defines the range of possible stress states of the structure. Resulting diagrams may be interpreted interchangeably as the form, or the forces in the corresponding form.

Airy stress functions are often referred to in the theory of shell structures [5, 9], in analysis and design, as a means of automatically satisfying horizontal equilibrium in a given shell structure. This paper makes

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use of so-called "2.5D" graphic statics where, using the Force Density Method [7] (FDM), 3D gridshell structures may be form-found for a given 2D form diagram (the flat projection of a gridshell), a known Airy stress function, and a known set of vertical loads. In this case, the 2D form diagram is both the projection of the gridshell structure as well as the projection of the Airy stress function, which are separate 3D forms which may but do not necessarily coincide [6].

In continuous shell structures, the Airy stress function can be categorised as a continuous surface. The stress components which describe equilibrium in the horizontal plane are the curvatures of the surface at any given point. Positive gaussian curvature implies pure tension or compression, such as in a fabric structure or in a masonry vault, respectively. Negative gaussian curvature, however, implies regions of both tension and compression. Understanding the stress state of the structure, therefore, is done by consideration of the principal directions of curvature of the Airy stress function, which coincide with the directions of principal stress in the loaded shell structure. In piecewise-linear (discrete) Airy stress functions, such as those describing loaded 2D trusses, curvature is concentrated entirely at the folds in the function which coincide with element locations in the 2D projection.

2. Pucher's Equation

As noted above, the Airy stress function is used here to satisfy horizontal equilibrium. The relationship between Airy stress function (φ) and shell geometry (F) for vertical equilibrium, for a shell under vertical loading (q) is given by Pucher's equation [8], stated below.

$$-q = \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 \varphi}{\partial y^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 \varphi}{\partial x \partial y} + \frac{\partial^2 F}{\partial y^2}\frac{d^2 \varphi}{\partial x^2} \tag{1}$$

A solution to this equation ensures equilibrium is achieved by membrane forces alone, and depends on the shape of the shell and boundary conditions. The existence of a solution is not guaranteed, and if one does not exist, bending action to some degree will also be required to resist load. The challenges around obtaining solutions are described in various sources [9, 11].

A key observation to be made on the Pucher's equation above is the interchangeability of shell geometry and Airy stress function [6, 10]. This allows for a known Airy stress function (φ) to be treated as a loaded shell (F), or vice versa. The identical process using FDM can be used on the known function of the pair (F or φ), to find the other. In effect, form-finding, which is solving Pucher's equation for F(x,y) (for a given set of loads and Airy stress function) is equivalent to solving for φ(x,y) (for a given set of loads and known shell geometry), and both are therefore possible using FDM. This paper makes extensive use of this in the studies presented, and element forces in the gridshell examples are determined in the process.

The Gaussian curvature of the Airy stress function (φ) affects the nature of Pucher's equations (1) when solving for F. Positive (synclastic surface) or negative (anticlastic surface) Gaussian curvature for φ result in Pucher's equation being elliptic or hyperbolic, respectively. Zero Gaussian curvature in φ results in Pucher's equation being parabolic. Note that the solution for F, the gridshell geometry, is not limited by the curvature of the Airy stress function, and so a hyperbolic φ might have an elliptic solution to F, etc. Stress functions may contain regions of all three types of Gaussian curvature. Solutions to Pucher's equation when it is hyperbolic or parabolic are challenging analytically [9,10].

3. Cantilevering Structures

Cantilevering shell and gridshell structures are an interesting structural typology that have not been given significant consideration to-date, in literature related to shell design, form-finding or to graphic statics. Form-finding tools and studies tend to consider structures either purely in tension or in compression. Cantilevering structures require regions of compression and tension, and therefore for an Airy stress function φ to contain negative Gaussian curvature, and for Pucher's equation therefore, to be hyperbolic in nature.

Miki et al. [11][12] proposes a computationally demanding method for solving Pucher's equation where both F and φ are treated as unknowns with known matching boundary conditions, and smooth continuous functions are obtained for both. As noted, this paper considers alternative methods to analyse and form-find cantilevering structures, using 2.5D graphic statics.

Figure 1: Autostadt Service Pavilion by Schlaich Bergermann Partner, © Tobias Hein, 2013

Despite the lack of research into form-finding methods of cantilevering structures, some notable examples of have been constructed. These include the Los Manantiales shell by Candela, the Zarzuela Hippodrome roof by Eduardo Torroja, the KnitCandela pavilion [14] by ETH Zurich and Zaha Hadid Architects, and others, which are examples of continuous shells with cantilevering regions. Examples of cantilevering gridshells appear to be rarer. The Autostadt Service Pavilion by Schlaich Bergermann Partner (Figure 1) is a cablenet structure which cantilevers from two sides.

4. Simple Cantilevering Structures

The simplest form of 3D cantilevering structure is a tetrahedral pyramid with its apex (referred to here as "free" node) projecting beyond the tetrahedron's base in plan and loaded vertically by a point load. The base nodes are assumed to be restrained vertically. The 2D projection of the tetrahedron is similar to Maxwell 1864 Fig.1, but with central free node dragged beyond its triangular boundary. This 2D projection has a single state of self-stress, corresponding to its vertically loaded 3D structure. The sign of the forces in the 2D projection's state of self-stress depend on the projected position of the free node and whether this sits within the boundary of the triangular base. This corresponds with the Airy stress functions for either of these cases obtained by "lifting" the free node.

In the examples below (Figure *2*) with a single free node, where therefore only a single lift is possible, deriving the Airy stress function is trivial and the examples must be of the "self-Airy" type. It is nevertheless useful to observe that the input 3D geometry can be used to construct a force diagram via graphic statics, the force density in elements determined from the resulting diagram, and FDM applied to derive the correct 3D output. The input and output may be considered as structural form and Airy stress function, respectively, or vice versa.

Note in the below that when the plan projection of the free node is moved beyond the base boundary, some planes of the Airy stress function move from the upper layer ("top") of the stress function viewed on plan, to the lower layer ("bottom"). Consideration of top and bottom layers is necessary for the correct determination of force sign when reading the Airy stress function, as force sign is observed to reverse in the respective layers.

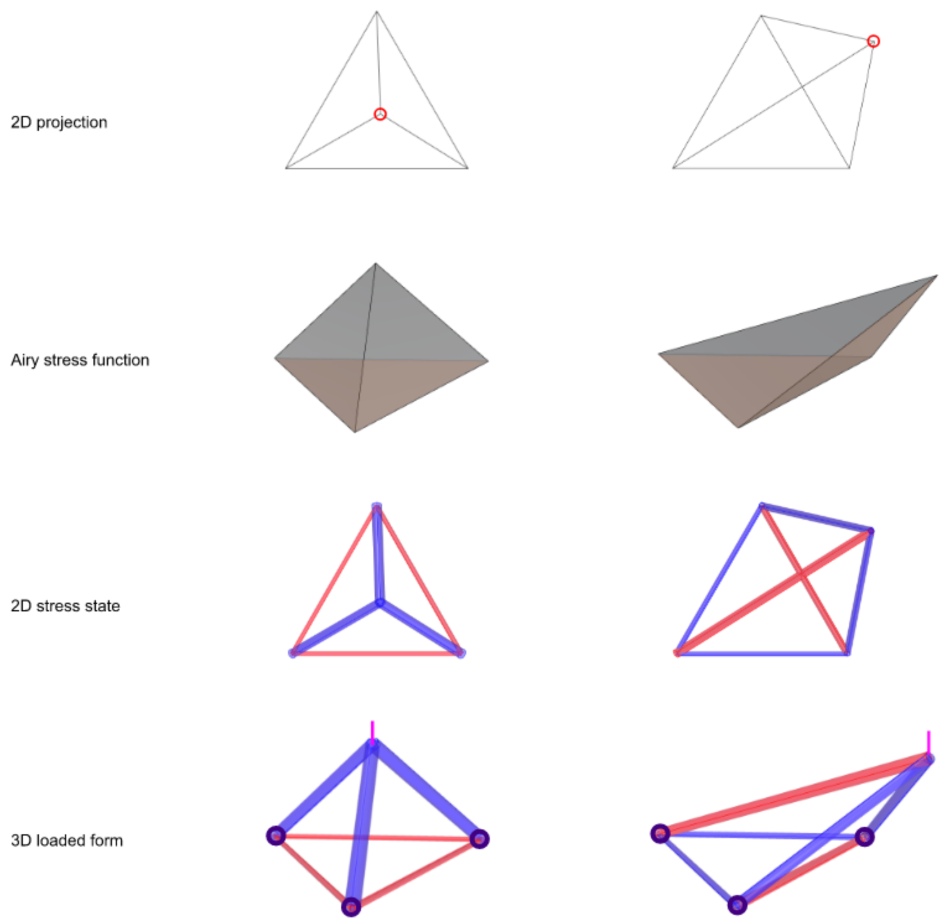

Figure 2: Top-to-bottom – row 1: left image similar to Maxwell 1864 Fig.1 while in right image "free" node (indicated in red) has been dragged beyond triangular boundary; row 2: 3D Airy stress function obtained by "lifting" free node, bottom surfaces indicated in orange; row 3: states of self-stress for either 2D projection, red indicating tension, blue indicating compression; row 4: 3D loaded structures with nodal load indicated in magenta and nodes supported vertically indicated in the base. 3D structure on the right is that of a cantilevering form.

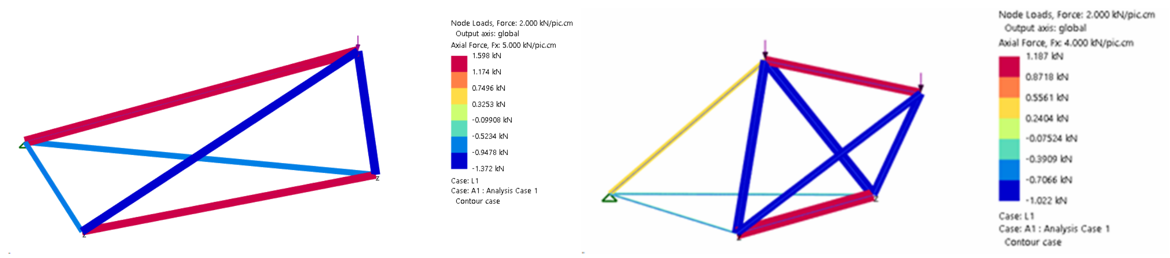

Figure 3: Verification in Oasys GSA showing similar axial load distribution as determined using proposed method.

Further similar simple constructions are presented below (Figure 4) with an increased number of free nodes, subject to vertical load. The process applied is identical to that above, with demonstrably correct results.

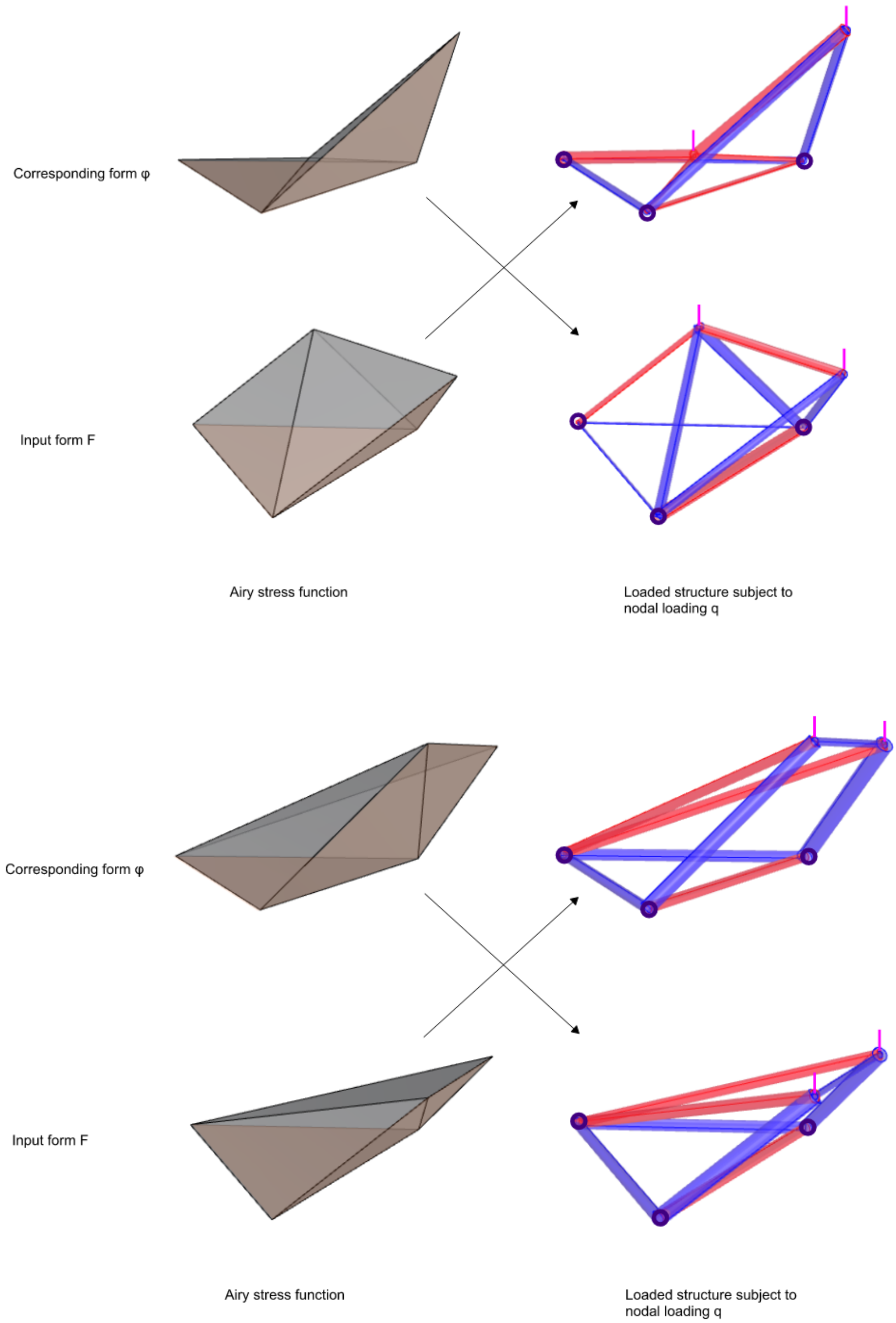

Figure 4: In either of these examples an input geometry F (bottom left) is treated as an Airy stress function, and used to derive forces, and find corresponding loaded structure (top right) of geometry φ, for given nodal loads indicated in magenta, using FDM. Geometry φ has in turn, been used as Airy stress function to derive forces in loaded structure of geometry F (bottom right) for given loads; tension is indicated in red and compression in blue, size of bars is indicative of relative force magnitude. Form-finding returns the input form. Vertical supports are indicated in images of loaded structures on the right (purple).

5. Verification of Results

Two different methods have been used in the verification of the form-finding process and its outcomes, respectively. The first is to use any output 3D geometry derived using FDM, as an input in an identical process (Figure 4). If the process is working correctly, it should return the original input form. The second method is simply to verify that any result is in fact in equilibrium and resists load predominantly (or entirely) axially, with a similar force distribution as that determined graphically, thereby confirming that the membrane solution to Pucher's equation identified is correct. This is done separately in finite element modelling package, Oasys GSA (Figure 3).

6. Discretised Continuous Forms

The process above is extended to more complex forms with significantly more nodes. The following construction is undertaken: a cantilevering curved surface representing a roof structure is drawn and its open and base faces are closed to create a solid volume; a triangular tessellation is applied to the volume surfaces, to create a closed mesh where mesh edges represent gridshell elements. The closed mesh is then used as an input in the process described above: 2D graphic statics is used to construct a force diagram and define force densities in elements; FDM is used to form-find the reciprocal 3D Airy stress function for a given set of vertical loads; this output is used to form-find the original input mesh (validating the results) along with associated forces in the original shell. Results from this process are presented below.

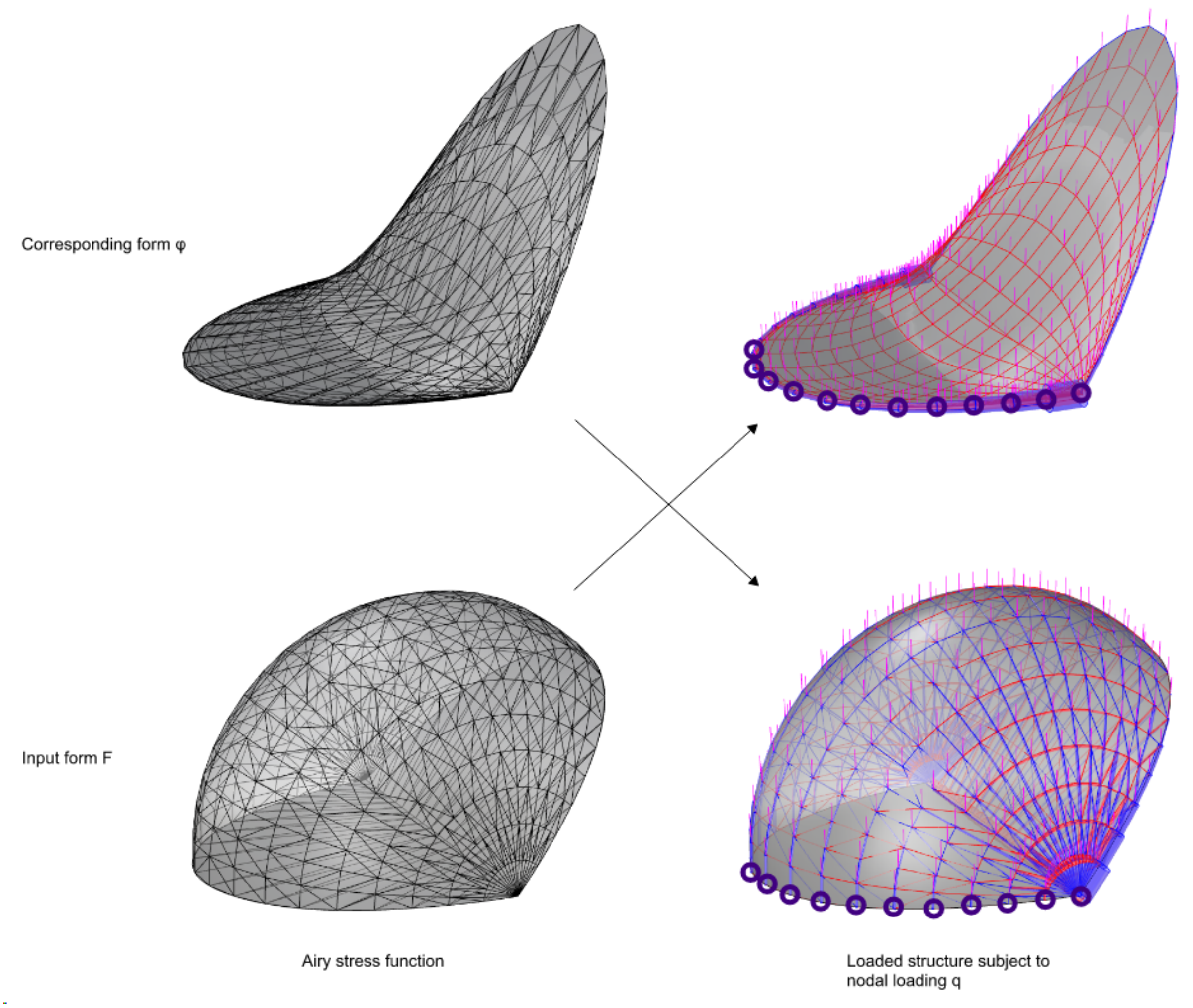

Figure 5: Example 1. Our initial input geometry F, treated as an Airy stress function (bottom left) to find the form of φ via FDM and its axial forces for given loads (top right). Considering φ as the Airy stress function (top left) returns the forces in the original input geometry F (bottom right).

6.1 Example 1: Volume with Single Cantilevering Surface

The first example (Figure 5) consists of an ellipsoidal surface that is incomplete so that it overhangs beyond the extent of its base. The open faces created by the surface are closed to create a solid volume, and a closed triangulated mesh of geometry F is created from this as described above. Fixed (providing vertical restraint) and free nodes are defined along with the vertical loading that applies at nodes. Top and bottom surfaces are also identified. Since the input geometry is synclastic, the output loaded structure of geometry φ can be expected to be entirely in tension or compression except at its boundary. This is confirmed in the resulting output, which resembles a cantilevering cable net structure with some geometric similarity to Autostadt Service Pavilion described above. Elements that result from the original triangular discretisation, but which contain no, or very low magnitude of force, are hidden in the images presented.

This output 3D geometry φ interpreted as Airy stress function describes the stress state of the original cantilevering convex gridshell (F). φ is observed to be a discretised anticlastic surface. This is to be expected as the original surface must contain regions of tension and compression to act as a cantilever. Note that the categorisation of whether nodes lie on the top or bottom surface of the Airy stress function is maintained from that in the input geometry, despite top and bottom intersecting in this case.

Forces in the original form are determined by constructing them using the output hyperbolic form as Airy stress function. FDM is also used to recover the original input shell, thus verifying the analysis.

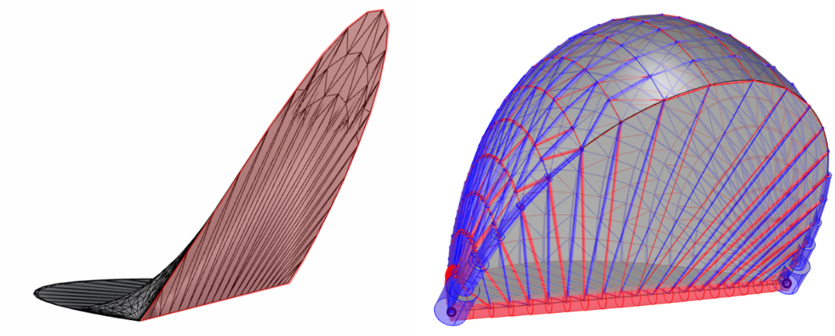

Figure 6: The elements in the Airy stress function lying on the surface indicated in red (left) are not co-planar. The corresponding elements in the loaded structure (right) must therefore be subject to axial forces.

It can be observed that the analysis shows that elements on the open face which are there due to the discretisation, are subject to axial forces (Figure 6). This is to be expected since the boundary in the Airy stress function φ does not lie on a plane. If those elements were to be removed, the Airy stress function indicates bending in the boundary element.

6.2 Example 2: Double Cantilevering Volume

The second example is again a convex surface though not based on hemispherical geometry and containing two opposite-facing cantilevering edges. The process and construction followed in example 1 is repeated here and results are presented below. The output loaded structure (of geometry φ) again resembles a cable net structure with cantilevering edges. The forces in the original input gridshell are determined and FDM is again used to return the original form and verify the analysis.

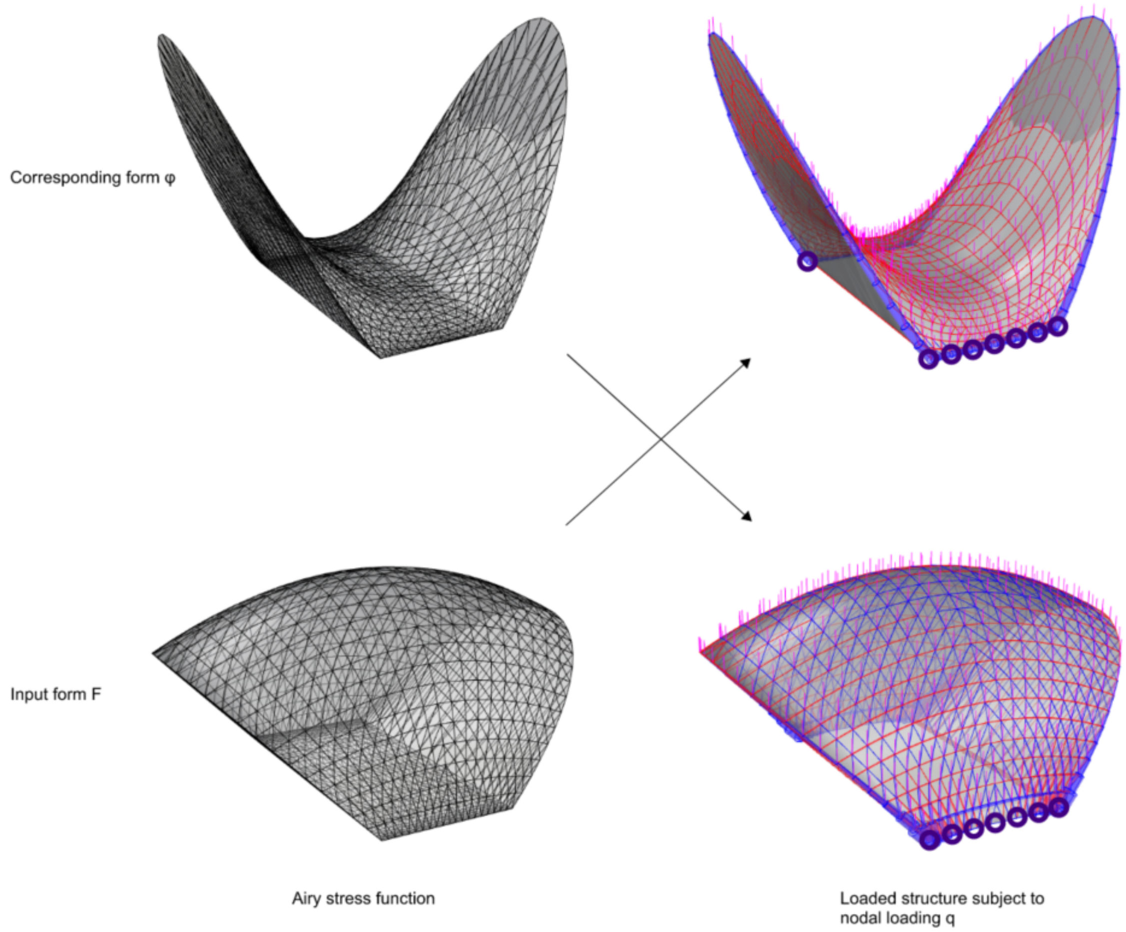

Figure 7: Example 2 follows an identical procedure to example 1 above with different input geometry F.

6.3 Manipulating the Airy Stress Function

In example 1, it was observed that the free edge of the cantilever of the original shell geometry must contain bending as the corresponding edge in the Airy stress function does not lie on a plane. It was therefore investigated whether the Airy stress function could be adjusted by tweaking the node positions to eliminate the bending and use FDM to form-find an alternative shell geometry. The Airy stress function proved to be extremely sensitive, however, and returned drastically different, highly irregular forms without any smoothness. It was found that even trying to rebuild a continuous surface from the Airy stress function and re-discretising to create a new, slightly different function yielded outputs which were drastically different from the original shell form (Figure 8). Despite the irregularity and lack of smoothness of the output, the results are demonstrated to be correct.

This observed sensitivity is due to the underlying surface we are discretising being hyperbolic, where elements close to asymptotic lines on the surface may require large forces to resolve vertical load, resulting in output geometries with large peaks. This may be limited by using discretisations that follow principal surface curvatures and avoid asymptotic lines. It is interesting to note that the proposed method is not subject to analytical challenges with boundary conditions associated with solving hyperbolic equations which are noted in [10, 11].

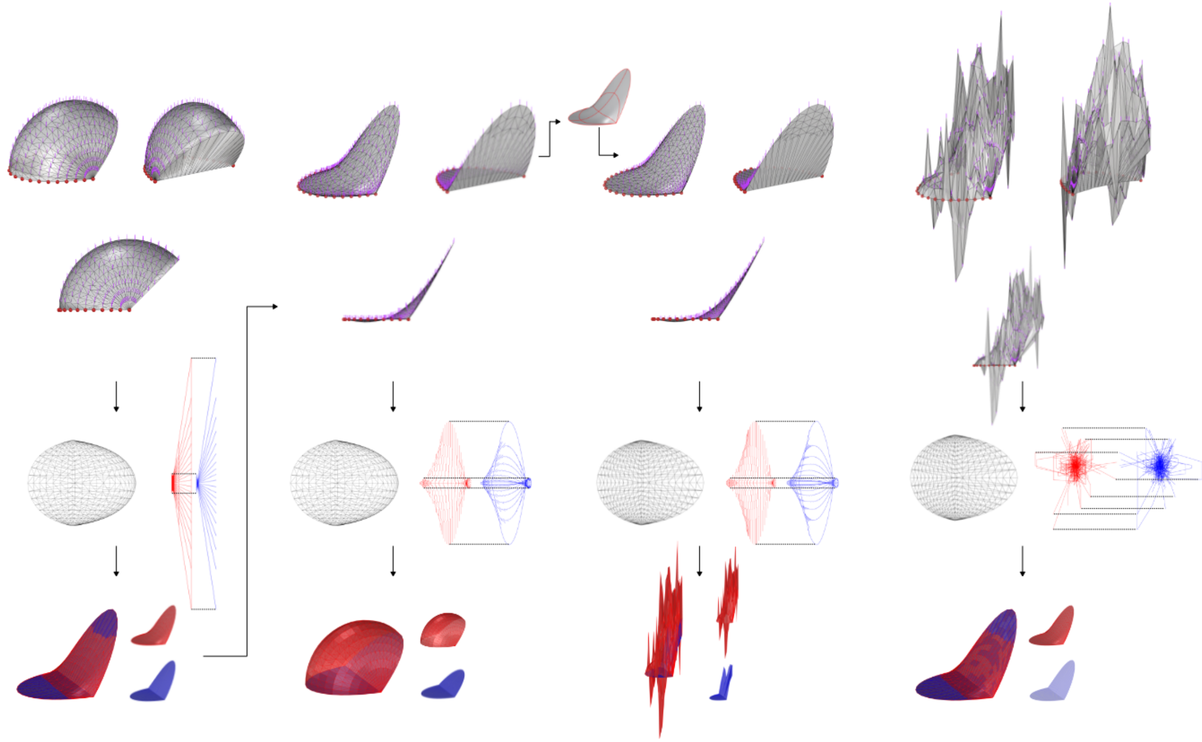

Figure 8: Top-to-bottom – row 1: loaded gridshell structures; row 2: flat projections of gridshells (left) and force diagram obtained by treating loaded gridshell in row 1 as Airy stress function; row 3: resulting Airy stress function found using FDM, identifying top (red) and bottom (blue) surfaces. Left-to-right – column 1: original loaded gridshell (top) example 1 is used to find associated Airy stress function (bottom); column 2: the procedure is repeated using output for Airy stress function from col. 1 to return the original input (bottom); column 3: the procedure is repeated using a geometry similar to that in col. 2 but with a slightly different surface discretisation, and while the force diagrams (row 2, cols. 2 and 3) are observed to be similar, the output geometry obtained using FDM is completely different and non-smooth; column 4: the procedure is repeated with the non-smooth output from col. 3 and returns the input used in col. 3 thus confirming the result.

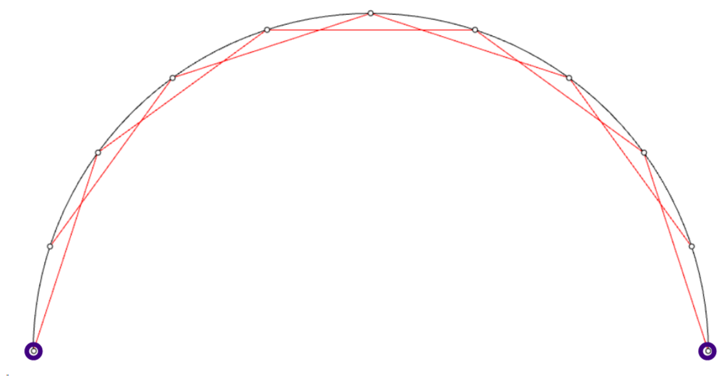

Figure 9: Alternative discretisation proposed for open faces of cantilevering gridshells which is demonstrated to achieve structural rigidity.

7. Structural Rigidity

A convex polyhedron is known to be rigid through Cauchy's theorem [13], provided that every face in the polyhedron is also rigid in its own plane. The discretised input geometries in the examples 1 and 2 discussed above, require their "open" lateral faces to be made rigid, for Cauchy's theorem to apply. It can be seen that this is achieved via the triangulated discretisation of these faces explained above. Other discretisations of these faces are also possible, that would allow for more open area while achieving rigidity. One such example is presented in Figure 9. The rigid structures can resist nodal loading solely via membrane action not just applied vertically, but in any direction.

Rigidity can be demonstrated by constructing the equilibrium matrix for the structure, and confirming the absence of mechanisms by performing singular value decomposition on the equilibrium matrix. (Without the additional bracing, the shell can be demonstrated to be subject to inextensional deformation at the main arch of the open face.)

8. Summary and Conclusions

This paper has presented a range of studies demonstrating the applicability of 2.5D graphic statics to cantilevering gridshells geometrically constructed by discretising continuous surfaces. A loaded gridshell may be understood both as 3D structural geometry and Airy stress function, which can be used to form-find a corresponding 3D geometry which is itself a loaded gridshell and Airy stress function, in a reversible procedure. Resulting output geometries are shown to be valid by independently demonstrating load paths and axial action using FEM. The effects of manipulating an Airy stress function have also been considered. The methods proposed are shown to be appropriate for cantilevering structures, despite the hyperbolic nature of the underlying equations of the continuous case.

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