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Graphical and numerical method of form-finding for membrane structures

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Abstract

This paper presents an intuitive method for form-finding of membrane structures using graphic statics. In the design of membrane structures, a higher degree of intricacy often results in complex geometric constraints, requiring a sophisticated design tool. In this paper, we extend the geometric method of finding edge cable radii developed by Frei Otto through reciprocal diagrams of form and forces. The proposed method allows for a precise control of edge cable forces and reaction force vectors of uniformly prestressed membrane structures. Our second contribution lies in the application of the force density method to planar isotropic stress fields and its reciprocal construction. We provide a graphical and a numerical proof of the duality between the two stress formulations based on nodal forces and natural forces. Lastly, through practical examples we demonstrate the application of the proposed method, in which the precise handling of the internal forces and reactions are required for project specific design constraints.

Keywords: Graphic statics, reciprocal figures, force density method, form-finding, membrane structure, conceptual design.

1. Introduction

1.1. Motivation

The finding of equilibrium for membrane structures is a well-developed topic, from the pioneering work of Frei Otto to computational modelling methods using various formulations. In the former, scaled down prototypes are utilized to develop understanding of form and fabrication, which is intuitive and sensible but can lack efficiency and precision [1]. In the latter, despite being fast and precise, one may soon lose track of the interplay between form and force, buried in the formulation of elements or the rather abstract task of defining force densities.

We are therefore motivated to bring the task of membrane form-finding under focus again, primarily through the lens of graphic statics. We present an intuitive method of finding equilibrium of cable reinforced membranes, by starting with a planar construction of reciprocal diagrams, followed by discretization and lifting of the 2D diagram through thereby informed force densities. Possible application of the method is shown by a few practical examples.

1.2. Related work

The force density method (FDM) was developed initially for cable net structures [2] such as the 1972 Olympic stadium roof in Munich, which linearly solves the network geometry by directly defining force densities of links in the net, given external loads and positions of anchor points, based solely on equilibrium conditions.

The extension of the force density method to tackle membrane form-finding, i.e. finding geometry of minimal surfaces given boundary geometries, was done by Singer [3], Maurin & Motro [4] and Pauletti [5] among others. Additionally, the idea of lumping stress of a triangular face membrane element can be traced in their work, which inspired the use of reciprocal figures in this paper. While Singer, Maurin and Motro proposed to lump stress as nodal forces which are perpendicular to opposite sides of a triangular face, Pauletti formulated the forces in the direction of the triangle edges hence the name "natural force density". We prove the equivalence of these two groups of methods, which is discussed further in Section 3.

Thrust network analysis (TNA) proposed by Block [6] was initially intended for compression-only masonry vaults and was extended for membrane structures [7], where the major task is finding equilibrium of a statically indetermined network given external loads, to approach a desired geometry by minimizing the differences in vertical positions with optimization techniques.

1.3. Scope

The scope of the work in this paper is limited to cable-reinforced, uniformly prestressed membranes, i.e. without stiff edge or ridge beams and with equal prestress in the fabric's wrap and weft directions.

No external loads are considered.

2. Planar equilibrium

2.1. Graphic statics and reciprocal figures

Maxwell proposed the concept of reciprocal figures i.e., a form diagram and a force diagram, to express the equilibrium of planar trusses [8], where the forces acting at one node in the form diagram is denoted by one closed polygon in the force diagram. The form and force diagrams can be constructed to any constant angle. Typically used are parallel (Cremona convention) and perpendicular (Maxwell convention) constructions. The latter has the property of preserving relative position of edges and was found suitable in the application presented in this paper (see Fig.1).

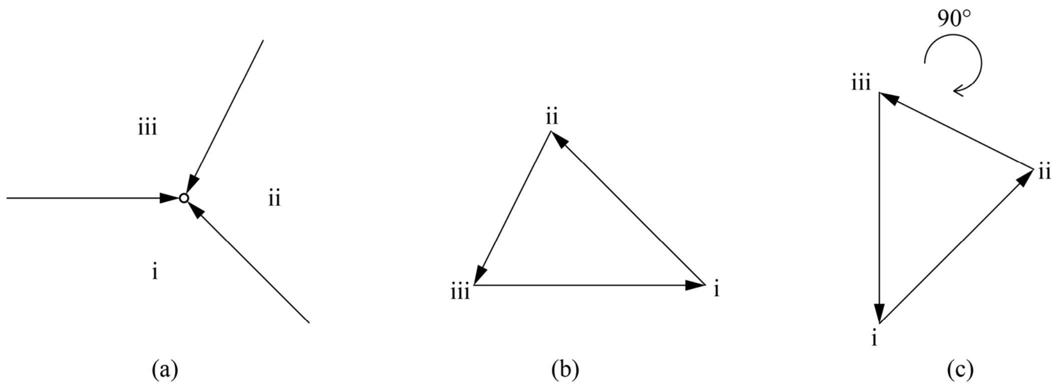

Figure 1: (a) Form diagram in which three forces act on a single point. (b) Force diagram using Cremona convention, in which forces are parallel to the corresponding edges in the form diagram. (c) Force diagram using Maxwell convention, in which forces are perpendicular to form edges.

2.2. Frei Otto's sketch on determination of anchoring force

The study of convertible roofs by Frei Otto and the IL team [9] documented a graphical method of determination of anchoring force direction in plane. This method indicates that the resultant force of two edge cables at this anchor point lies in the direction of shared chord of the two circular profiles of the cables. This geometric property can be further explored with the aid of reciprocal figures of form and force and the well-known formula $F = p \cdot r$ for hydrostatic pressure. The reciprocal figures in Figure 2 illustrate the nodal equilibrium at the intersection of two edge cables of two different radii. The hydrostatic pressure provides a linear relation between the edge radius and the cable force, where the direction of the cable force can be drawn perpendicular to the circular edge in the force diagram (green and blue lines). Consequently, the anchoring force as the sum of two edge cable forces $F_1 + F_2$ (in red) lies in the direction of the shared chord and has the amplitude of the distance between two circle centers provided that the prestress is unit isotropy. Here, the form edge, i.e. anchoring cable, and the force edge, i.e. connection of two centers, sit orthogonally to each other following the Maxwell convention.

This principle of constructing nodal equilibrium of cable ends can be extended to the entire perimeter of a cable supported membrane structure, resulting in a closed force polygon that describes the equilibrium of anchor forces, as illustrated in Figure 3.

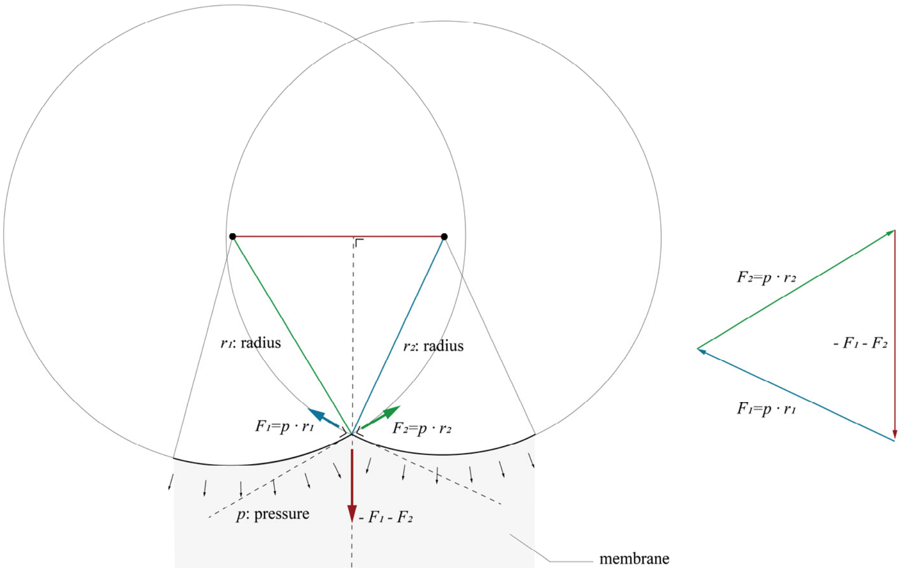

Figure 2: (left) Anchoring cable and anchoring force is a reciprocal pair following Maxwell convention. (right) Rotated figure in Cremona convention for clarification.

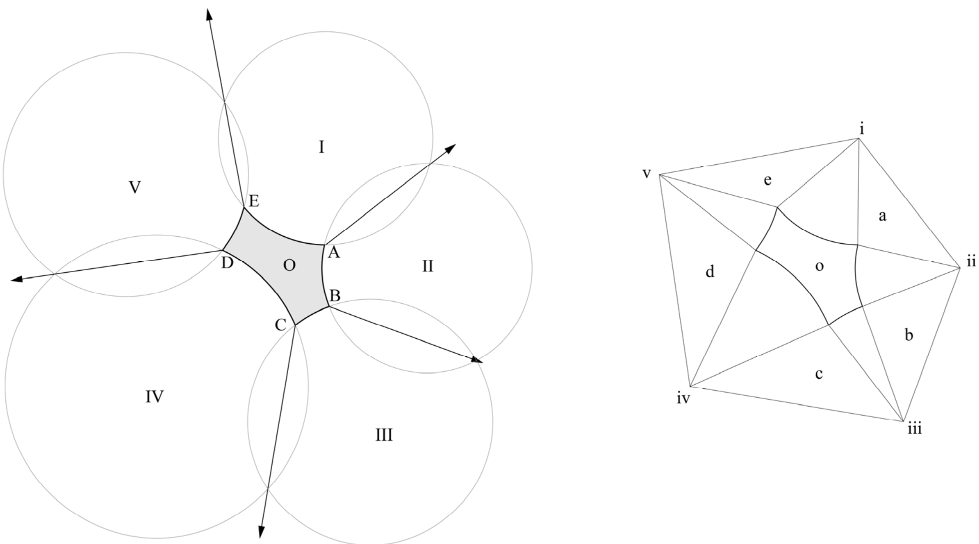

Figure 3: (left) Frei Otto's drawing method applied to a 5-supports membrane structure, creating a form diagram. (right) Corresponding force diagram following the Maxwell convention. The outer closed polygon indicates the equilibrium of all anchor forces. The triangles a-e represents the nodal equilibrium of nodes A-E.

3. Vertical equilibrium

3.1. Discretization: lumping isotropic stress as nodal forces

Section 2 discussed horizontal equilibrium of isotropic membrane structures in a two-dimensional plane using graphical construction of the edge cable forces. The continuous isotropic stress field requires discretization in order to be used to find three-dimensional vertical equilibrium based on the found two-dimensional equilibrium. Using the shape function of constant strain triangular (CST) elements [10], the elementwise kinematic matrix can be expressed as follows.

$$B = \frac{1}{2S}\begin{bmatrix} y_C - y_B & 0 & -y_C + y_A & 0 & y_B - y_A & 0 \\ 0 & -x_C + x_B & 0 & x_C - x_A & 0 & -x_B + x_A \\ -x_C + x_B & y_C - y_B & x_C - x_A & -y_C + y_A & -x_B + x_A & y_B - y_A \end{bmatrix} \tag{1}$$

Here, x and y denote the coordinates of node A, B and C, and S the area of the triangle (see Fig. 4). Using the static-kinematic duality $A = B^T$ with A being the elementwise equilibrium matrix, the nodal force $n = [n_{A,x} \quad n_{A,y} \quad n_{B,x} \quad n_{B,y} \quad n_{C,x} \quad n_{C,y}]$ can be expressed with respect to the element force $\sigma = [\sigma_1 \quad \sigma_2 \quad \tau]$ as follows.

$$n = SA\sigma \tag{2}$$

With the assumption of a unit isotropic stress field $\sigma = [1 \quad 1 \quad 0]$, the nodal force yields the following:

$$n = \frac{1}{2}[y_C - y_B \quad -x_C + x_B \quad -y_C + y_A \quad x_C - x_A \quad y_B - y_A \quad -x_B + x_A]^T \tag{3}$$

$$[n_A^T \quad n_B^T \quad n_C^T] \tag{4}$$

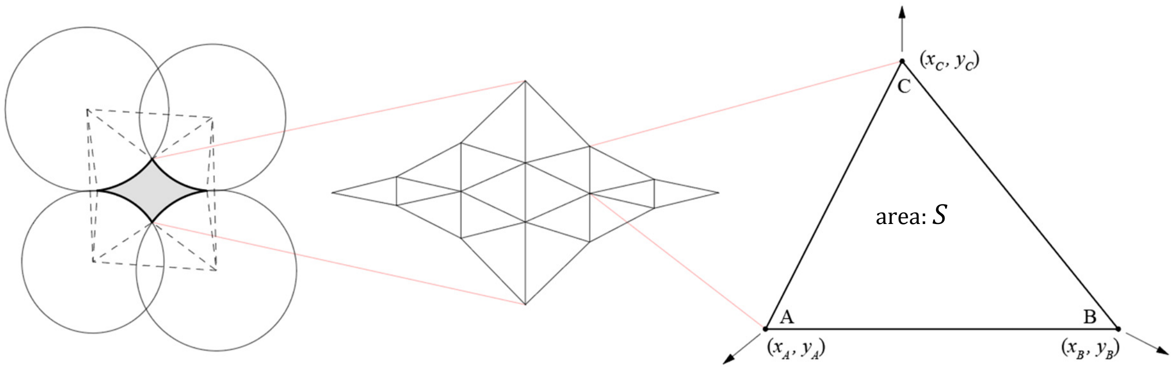

Figure 4: Discretization of an isotropically stressed membrane patch reinforced by edge cables

We refer to these derived forces in n_A, n_B, n_C as “nodal forces” in this paper. By looking at the expression (3), two observations can be made: (i) the nodal force is perpendicular to the opposing edge, and (ii) the length of the nodal force is half of that of the opposing edge. These lemmas can be easily proven by using the edge vector $u_{BC} = [x_C - x_B \quad y_C - y_B]$, with which the orthogonality $n_A \cdot u_{BC} = 0$ and the length identity $|n_A| = |u_{BC}|/2$ hold valid. As mentioned, different approaches regarding these properties of isotropic membrane stress were discussed by Singer [3] and Maurin & Motro [4].

3.2. Equivalent linear element and reciprocal figure of form and force

With these two lemmas, we can draw a reciprocal figure of *form* and *force* polygon as shown in Figure 5. The form diagram is made by introducing an equivalent linear element, a line from each vertex (A, B, C) to the orthocenter D, while the force diagram consists of edges orthogonal to corresponding nodal forces, which fits exactly inside the triangular element, i.e. the edges of the median triangle.

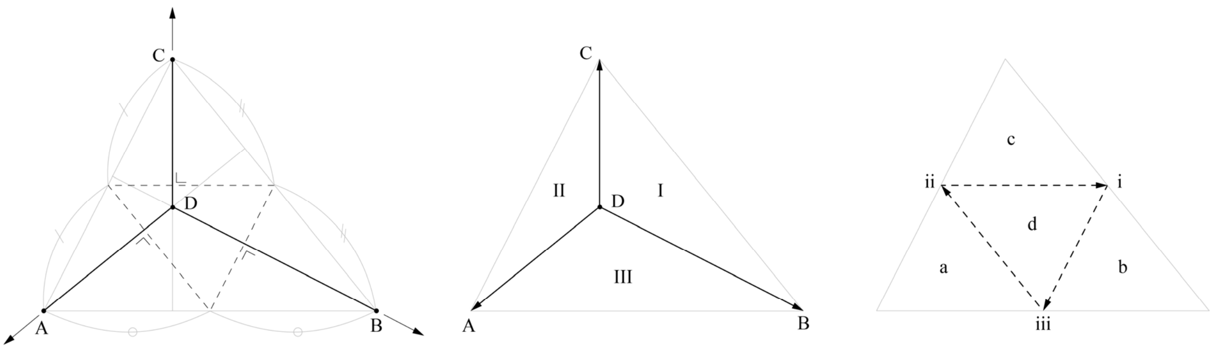

Figure 5: Form and force diagram of an isotropic CST element with help of additional node D, the orthocenter of the triangle. (left) superposition of the two diagrams, (middle) form diagram and (right) force diagram.

One can extend the elementwise diagram to the reciprocal figure of form and force polygon of an entire 2D isotropic membrane structure as illustrated in Figure 6.

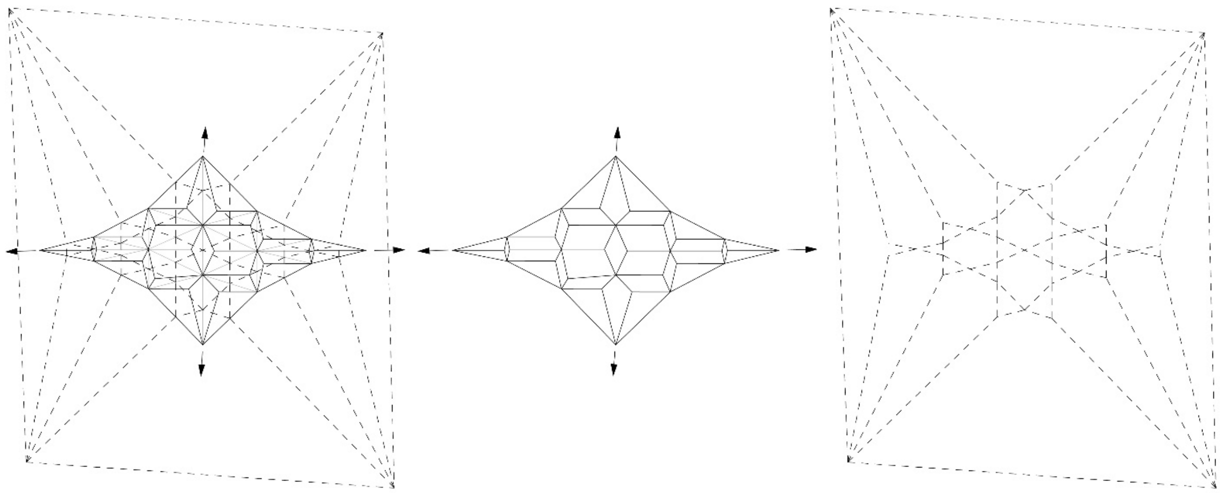

Figure 6: Form and force diagram of a 2D isotropic membrane structure with edge cables: (left) superposition of the two diagrams, (middle) form diagram and (right) force diagram.

The constructed 2D equilibrium can then be used as a basis for finding a vertical equilibrium using the force density method (FDM). However, this method has potential drawbacks when used in conjunction with the FDM. Firstly, the new node (orthocenter) signifies a new degree of freedom, increasing the total degree of freedom and thus the computational cost. Secondly, one edge of the form diagram vanishes in right-angled triangles, and one edge direction flips in obtuse triangles, rendering the force density value of the edge negative (see Fig. 7).

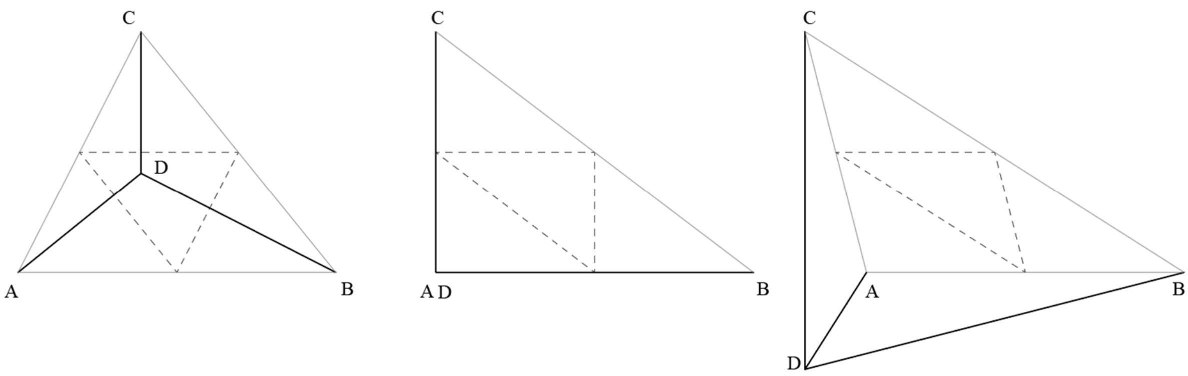

Figure 7: Form and force diagram of an acute (left), right-angled (middle) and obtuse (right) triangle.

3.3. Reciprocal figure of form and force using natural force

The drawbacks discussed in Section 3.2 can be mitigated using natural forces instead of nodal forces which have been discussed so far. Natural forces represent forces in a membrane element which are oriented in the direction of the element edges [5]. The reciprocal figure reveals that a nodal force acting on one node can be decomposed by natural forces acting on two connecting edges, while retaining the orthogonality of the form and force diagram. Figure 8 illustrates the superposition of the reciprocal figure for nodal forces (in red) and natural forces (in blue). Furthermore, an interesting duality is revealed here: the form diagram of the nodal forces is similar in shape to the force diagram of the natural forces, and the force diagram of the nodal forces is similar in shape to the form diagram of the natural forces.

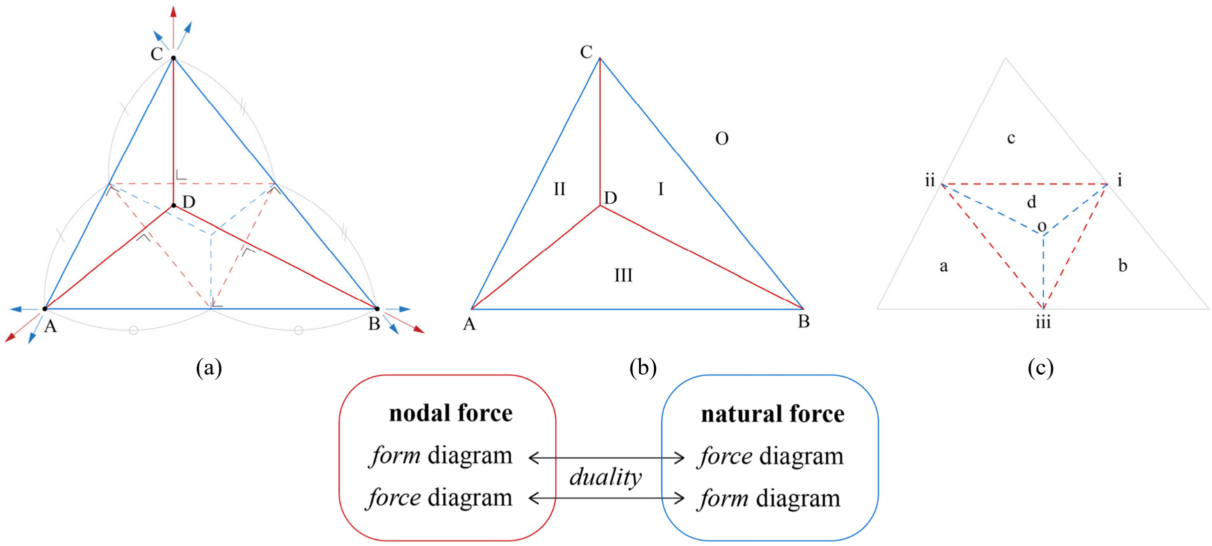

Figure 8: (a: left) Superposed diagrams of (b) and (c). (b: middle) Superposed form diagrams of nodal forces and natural forces. (c: right) Superposed force diagrams of nodal and natural forces.

3.4. Algebraic proof for the identity between nodal and natural force diagram

This graphical duality of nodal and natural forces leads to the following algebraic identity, where the force density of the nodal force is the inverse of the force density of the natural force.

$$q_C = \frac{f_C}{l_C} = \frac{l_{AB}}{f_{AB}} = \frac{1}{q_{AB}} \tag{5}$$

Here, f, l and q denote respectively the force, the length and the force density of each diagram (see Fig. 8). The equilibrium matrix of the triangular element (e.g. for z direction) with respect to nodal forces can be expressed using connectivity matrix C and diagonal matrix of the force densities Q as follows:

$$K = C^T Q C = \left[\begin{array}{ccc|c} q_A & 0 & 0 & -q_A \\ 0 & q_B & 0 & -q_B \\ 0 & 0 & q_C & -q_C \\ \hline -q_A & -q_B & -q_C & q_A + q_B + q_C \end{array}\right] = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \tag{6}$$

In this expression, the fourth degree of freedom represents the node D. Performing static condensation on the node D, the reduced equilibrium matrix with respect to nodes A, B and C can be written as follows.

$$K_{red} = K_{11} - K_{12} K_{22}^{-1} K_{21} = \frac{1}{q_A + q_B + q_C} \begin{bmatrix} q_A q_B + q_A q_C & -q_A q_B & -q_A q_C \\ -q_A q_B & q_A q_B + q_B q_C & -q_B q_C \\ -q_A q_C & -q_B q_C & q_A q_C + q_B q_C \end{bmatrix} \tag{7}$$

With the identity $q_c = 1/q_{AB}$, the reduced equilibrium matrix K_{red} can be rewritten with respect to natural forces as follows.

$$K_{red} = \begin{bmatrix} q_{AB} + q_{CA} & -q_{AB} & -q_{CA} \\ -q_{AB} & q_{AB} + q_{BC} & -q_{BC} \\ -q_{CA} & -q_{BC} & q_{CA} + q_{BC} \end{bmatrix} \tag{8}$$

The equilibrium matrix (8) represents precisely the element matrix of the natural force density method (NFDM) proposed by Pauletti et al [5].

3.5. Solving 3D equilibrium

Using either a nodal or a natural force diagram, the FDM, or the NFDM in the latter case, can be constructed for finding the vertical equilibrium. Figure 9 (left) illustrates the reciprocal diagram using natural forces for the complete structure, while Figure 9 (right) shows its ‘lifted’ three-dimensional equilibrium for the elevated support points in space. The obvious advantage of this approach is that the initial 2D diagram being the projection of the 3D structure allows for precise plan arrangement of the structure without resorting to a complex numerical computation. This lifting method based on the force density approach has a side effect: the resulting forces in 3D are scaled by the amount of slope and as this slope is variable across the surface, the result is an anisotropic stress state in 3D.

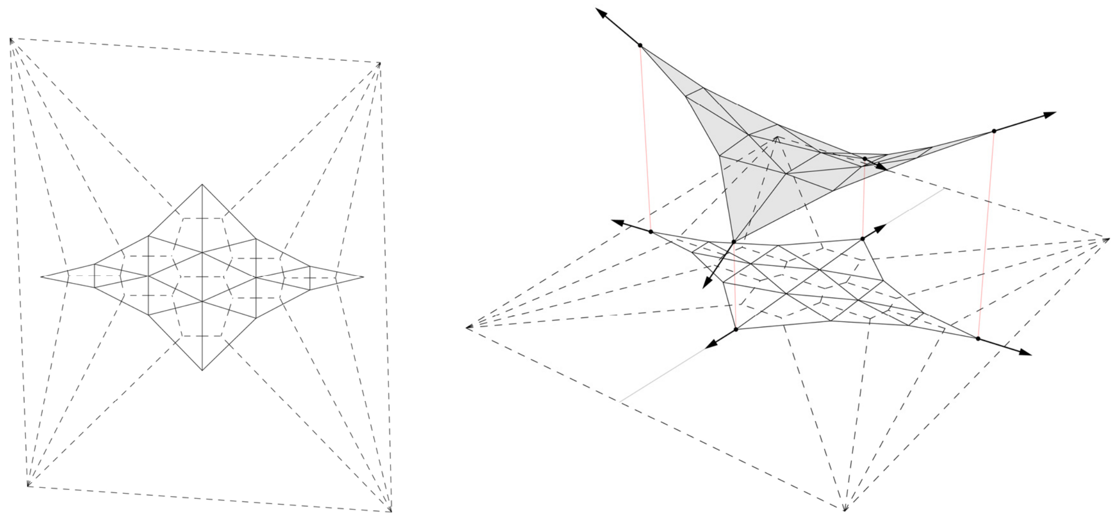

Figure 9: (left) Superposed reciprocal figure of form and force diagram using natural forces. (right) ‘lifted’ three-dimensional equilibrium as a result of the NFDM.

4. Applications

4.1. Convertible membrane roofs

In the planning of a convertible membrane roof, the design of moving anchors on dedicated rails plays a crucial role. Mainly two strategies can be observed, firstly with sliding anchors on low-friction rails (passive anchors) pulled by e.g. a driving cable and secondly using motorized anchors with gears on linear racks (active anchors). In the former case the anchors transfer only perpendicular forces to the substructure and have no stiffness along the rail, while in the latter the anchors rigid along the rail is capable of transferring inclined reactions , at the cost of more motors and advanced rack design. For built applications, the authors refer to the roof over a street canyon in Buchs, Switzerland, and the city hall roof in Vienna, Austria [11]. In either case, a minimized longitudinal force component along the rails would be desired, which can be planned in a straightforward manner with the method discussed.

Given a site plan of an unparallel street canyon, the planar anchoring forces can be determined graphically by sketching out a series of edge cable arcs. In the following example, ridge cables which span through the membrane are present, which introduce new form and force edges to the reciprocal diagrams, following the orthogonal construction principle.

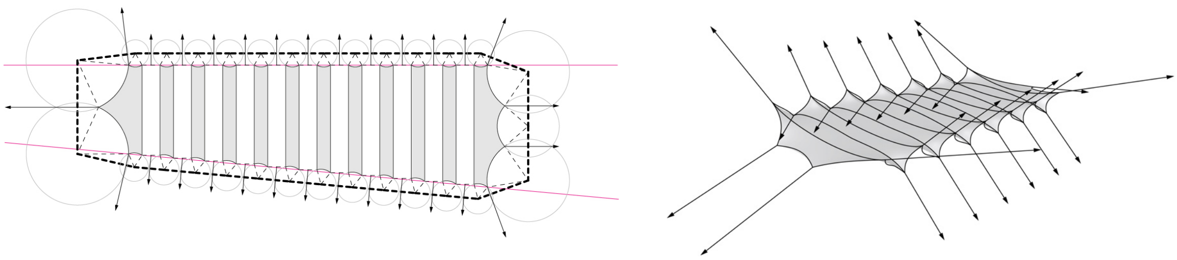

Figure 10: (left) 2D form and force diagram of the membrane structures. Arrows indicate the direction and the magnitude of the reactions, with their force edges in Maxwell convention forming a closed polygon. (right) Lifted 3D equilibrium made using the proposed variation of the force density method.

4.2. Cable supported membrane roof of complex support conditions

Figure 11 illustrates a cable supported membrane structure, characterized by a complex plan arrangement consisting of a circular part and a straight part that are joined at the center. Due to the orientation of the supporting structure, the directions of the resultants for supporting cables are fixed in the radial direction, which results in variable radial cable forces as indicated by the size of the circles. In the straight part, all the support directions are fixed in the perpendicular direction to the path again due to the nature of the supporting structure. The proposed graphical method allowed to effortlessly solve this complex geometric and support constraints in the projected two-dimensional plane, which was then lifted to the three-dimensional equilibrium using the proposed variation of the force density method.

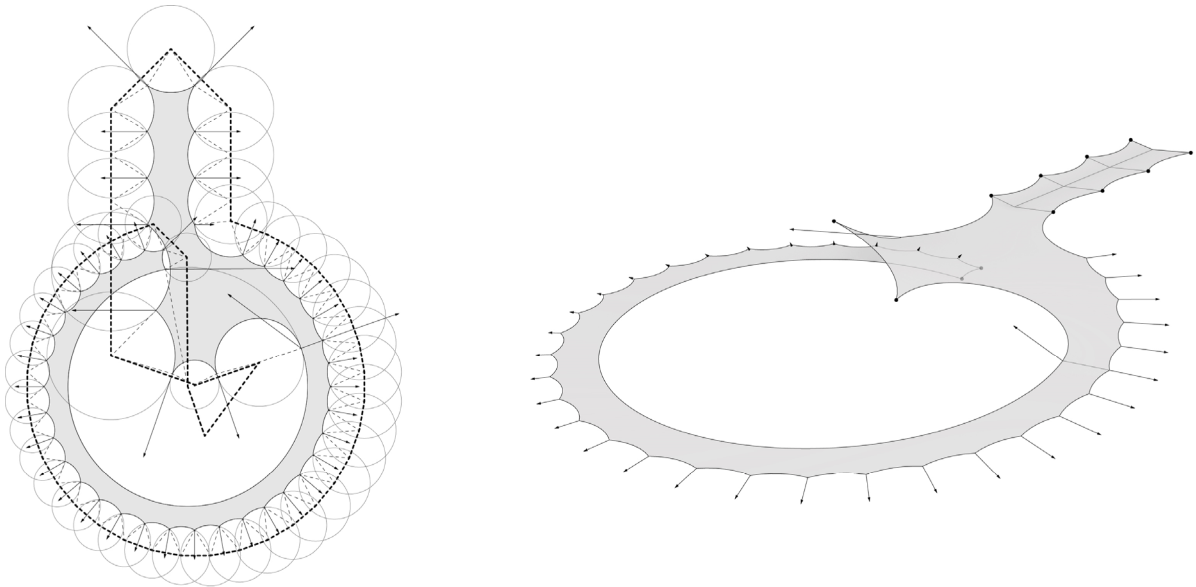

Figure 11: (left) 2D form and force diagram of the membrane structures. Arrows indicate the direction and the magnitude of the reactions, with their force edges in Maxwell convention forming a closed polygon. (right) Lifted 3D equilibrium made using the proposed variation of the force density method.

5. Conclusion

This paper introduced a graphic approach to understanding and solving 2D and 3D equilibrium of cable supported, uniformly prestressed membrane structures. Frei Otto's drawing rule was extended using reciprocal figures of form and forces, leading to a more comprehensive method to draw 2D equilibrium of membrane structures. Furthermore, the isotropic stress state of triangular membrane elements was described using graphical construction, providing the equivalence between two different approaches based on nodal and natural forces. This discretization method allows for a straightforward application of the force density method to membrane elements, which preserves anchor force directions of the 2D equilibrium when lifted into 3D. These toolsets facilitate an integrated approach in the design of membrane structures, in which complex project specific geometric constraints can be addressed.

6. Discussion and future work

A drawback of the proposed method lies in the scaling of forces when lifted from 2D to 3D structures. The starting assumption of 2D isotropic stress results in anisotropic stress in 3D due to variations in the produced slopes. Isotropic stress can be achieved by updating forces based on the actual geometry several times, similarly to the updated reference strategy (URS) and the NFDM [5]. However, this iterative procedure loses the important feature of the preservation of anchor force directions. Approaches to address this issue are worth exploring in the future.

Furthermore, this paper limited the graphical representation of membrane forces to isotropic stress states. The representation of anisotropic stress states using reciprocal figures is another point of interest for further investigation.

Acknowledgement

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References

[1] L. Gruendig, E. Moncrieff, P. Singer, and D. Ströbel, 'A history of the principal developments and applications of the force density method in Germany 1970–1999', Jan. 2000.

[2] H.-J. Schek, 'The force density method for form finding and computation of general networks', *Computer Methods in Applied Mechanics and Engineering*, vol. 3, no. 1, pp. 115–134, Jan. 1974, doi: 10.1016/0045-7825(74)90045-0.

[3] P. Singer, 'Die Berechnung von Minimalflächen, Seifenblasen, Membrane und Pneus aus geodätischer Sicht', Diss., Universität Stuttgart, Institut für Anwendungen der Geodäsie im Bauwesen, 1995.

[4] B. Maurin and R. Motro, 'The surface stress density method as a form-finding tool for tensile membranes', *Engineering Structures*, vol. 20, no. 8, pp. 712–719, Aug. 1998, doi: 10.1016/S0141-0296(97)00108-9.

[5] R. M. O. Pauletti and F. L. Fernandes, 'An outline of the natural force density method and its extension to quadrilateral elements', *International Journal of Solids and Structures*, vol. 185–186, pp. 423–438, Mar. 2020, doi: 10.1016/j.ijsolstr.2019.09.003.

[6] P. Block, 'Thrust Network Analysis : exploring three-dimensional equilibrium', Thesis, Massachusetts Institute of Technology, 2009. Accessed: Dec. 09, 2022. [Online]. Available: https://dspace.mit.edu/handle/1721.1/49539

[7] T. Van Mele and P. Block, 'A novel form finding method for fabric formwork for concrete shells', *Journal of the International Association for Shell and Spatial Structures*, vol. 52, Dec. 2011.

[8] J. C. Maxwell, 'XLV. On reciprocal figures and diagrams of forces', *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, vol. 27, no. 182, pp. 250–261, Apr. 1864, doi: 10.1080/14786446408643663.

[9] Institut für leichte Flächentragwerke, *IL5 Wandelbare Dächer Convertible Roofs*. 1971.

[10] J. S. Przemieniecki, *Theory of matrix structural analysis*. New York: Dover, 1985.

[11] R. Bergermann, J. Gugeler, and T. Keck, 'Wandelbares Membrandach im Innenhof des Wiener Rathauses', *Stahlbau*, vol. 73, no. 6, pp. 373–380, Jun. 2004, doi: 10.1002/stab.200490105.