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Free-form coupled funicular curves

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Abstract

Funicular structures allow the design of highly material-efficient, slender structures. At the same time, the requirement of funicularity eliminates the designer's freedom in forming the structure, and limits potential areas of application. The apparent contradiction between free-form design and funicularity can be resolved by considering structures with complex topologies. This work focuses on 'coupled curves': structures composed of two, planar, curvilinear components connected by a sequence of coupling elements (cables, hangers, columns). Coupled curves can be considered as conceptual models of several bridge types and of beams with external post-tensioning. The traditional design approach to coupled curves is to choose the shape of one curve (e.g. the deck of a bridge) freely or as dictated by functional requirements, and to design the shape of the second one (e.g. pylon of a harp bridge) as dictated by funicularity. This paper explores a more advanced problem: the goal is to construct a set of coupling elements, which makes two planar curves with arbitrary prescribed shapes, and loads funicular. The problem is formulated as an initial value problem with the distribution of coupling members and internal forces in all members as unknowns. With appropriate choice of initial conditions, both endpoints of the curves can be prescribed by the designer. The proposed method makes it possible to shape the two main components of coupled curves freely, while preserving funicular state in all components.

Keywords: Form-finding, free-form design, funicular structures, external post-tensioning, initial value problem, boundary value problem

1. Introduction

Funicular structures balance their external loads via pure tension or compression forces, which allows the design of highly material-efficient, slender structures. However, the shapes of funiculars are strongly connected to the external loads. Aesthetic, functional, and technological requirements often contradict the requirement of funicularity, which leads to the design of structures under significant bending. By considering structures with complex topologies, it is possible to design structures that preserve funicular state in all members regardless of their shapes. Well-known examples include trusses, and network arch bridges. As we will show in this work, '*coupled curves*' – structures composed of two planar curvilinear beams and dense sequence of non-intersecting connection elements such as hangers, cables, or columns also enjoy this beneficial property, at least for one distribution of the loads.

Coupled curves can be regarded as conceptual mechanical models of several bridge types (e.g. arch, cable-stayed, harp, and suspension bridges), and beams with external post-tensioning. Our results therefore reveal new design methods for these structures, many of which are not only masterpieces of engineering design, but also iconic elements of natural or man-made landscapes and means of artistic expression. In case of the most popular bridge types there exists thorough literature on the optimisation of shape (Farshad [1], Zhang et al. [2]). Literature about the set of achievable shapes of those structures is a less studied topic, but as Zwingmann [3] explains, a topic of increasing significance.

A discrete version of form-finding of coupled funicular curves has been studied by Todisco et. al. Motivated by application to externally post-tensioned beams, those authors investigated the situation when the shape of a polygonal structural member is prescribed along with the directions of connection elements at the vertices [4], [5]. A graphic statics-based method was developed for the form-finding of the second polygonal member.

This work presents a different approach to the problem in two different aspects. Firstly, we consider a continuous limit, in which the connecting elements are dense, and the polygonal member becomes a smooth curve. Secondly, the goal of the present paper is to consider two members with prescribed shapes along with their loads, and to construct an appropriate sequence of connection elements, such that both curves become funicular. In addition, the proposed method allows the designer to prescribe the locations of the initial and terminal points on the two coupled curves. The significance of this approach lies in that the two curves are often the primary functional elements of these structures, which are also visually more prominent than the connectors.

From the equilibrium equations of infinitesimal pieces of the curves we obtain an initial value problem (IVP) composed of a system of three ordinary differential equations, with unknown functions representing the distribution of connection elements and internal forces in all members. The feasibility of this approach is illustrated by examples of numerical solutions produced by a MatLab-based solver. We also classify all mathematical issues, which occasionally lead to failure of the method.

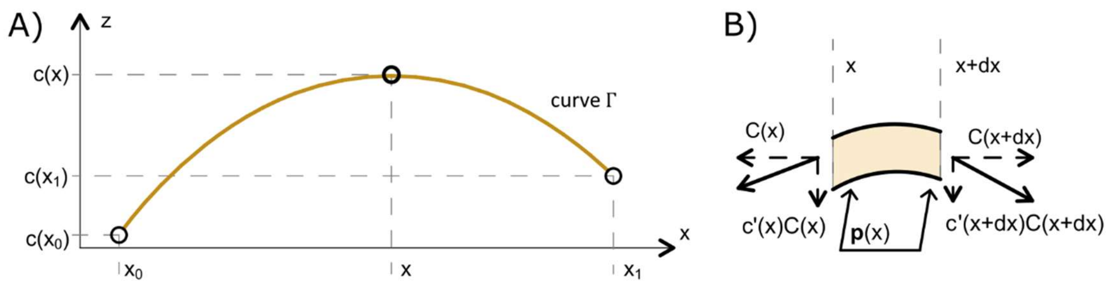

Figure 1: A) notation and parametrization of curve Γ; B) Free body diagram of an infinitesimal segment of curve Γ in funicular state.

2. Equilibrium equations of coupled funiculars

2.1. The concept of a funicular curve

Consider a Cartesian coordinate system xz and a curve $\Gamma\colon \mathbf{c}(x) = (x, c(x))$ parameterized by the x coordinate of its points, representing the axis line of a thin (curved) structural member. Let $\mathbf{p}(x) = (p_x(x), p_z(x))$ be a function representing distributed load of the structure. We say that a finite segment of curve Γ given by $x \in [x_0, x_1]$ is funicular for the load $\mathbf{p}$ if there exists a scalar function $C(x)$ such that any part of the curve is in static equilibrium under the external load $\mathbf{p}$, internal force function $\mathbf{N}(x) = [C(x), C(x)c'(x)]$ and two external forces including $C_0 = -[C(x_0), C(x_0)c'(x_0)]$ at point x_0 and $C_1 = [C(x_1), C(x_1)c'(x_1)]$ at point x_1 representing support reactions at the terminal points of the segment (Figure 1). Note that $\mathbf{N}(x)$ is everywhere tangential to Γ, and thus the force system outlined above corresponds to a curved rod in equilibrium under pure tension/compression. The equations of equilibrium of an infinitesimal piece of the rod yield the following conditions of funicularity [6]:

$$p_x(x) + C'(x) = 0 \tag{1}$$

$$p_z(x) + c''(x)C(x) + C'(x)c'(x) = 0. \tag{2}$$

For prescribed load $\mathbf{p}$, and endpoints $(x_i, f(x_i))$, (1) can be solved independently, and the solution has one free parameter (e.g. the initial value $C(x_0)$). Then, (2) becomes a linear boundary value problem with a unique solution in most cases. The one-parameter set of solutions leaves very limited freedom for the designer to take functional and artistic aspects of the problem into consideration. In addition, some solutions of (1) include a point x with $C(x) = 0$, while $p_z(x) \neq 0$. Then (2) is not solvable, and thus no corresponding funicular shape exists.

2.2. Coupled funiculars

One possible way to increase the freedom of the designer, and to resolve the contradiction between optimal structural form, and functional or artistic needs, is considering structures with more complex topology. We will consider structures composed of two curves: $\Gamma_a: \mathbf{a}(x) = (x, a(x))$, and $\Gamma_d: \mathbf{d}(x) = (x, d(x))$. Motivated by application in bridge design, these will be referred to as *arch* and *deck*. Both curves are subject to arbitrary loads $(\mathbf{p}_a, \mathbf{p}_d)$ and are required to be funicular. The two curves are connected by a dense sequence of load transferring elements referred to as *hangers* (both in case of compression and tension), which will be idealized as a continuous, weightless, distributed connection. The hangers are represented by a function $k(x)$ mapping point $\mathbf{d}(x)$ of the deck into the corresponding point $\mathbf{a}(k(x))$ of the arch (Figure 2), and by a vector-valued function

$$\mathbf{K}(x) = K(x)\frac{\left(\mathbf{a}(k(x)) - \mathbf{d}(x)\right)}{|\mathbf{a}(k(x)) - \mathbf{d}(x)|}, \tag{3}$$

representing distributed force exerted by the cables on the deck. Here $K(x)$ is the magnitude of the cable force, and the rest of the formula is a unit vector in hanger direction. Positive sign of $K(x)$ represents tension in the hangers. In the followings, we will refer to such a system as a pair of *coupled curves*. A segment $x \in [x_0, x_1]$ of a coupled curve is called *funicular*, if there exists a function $K(x)$, for which

- the segment $x \in [x_0, x_1]$ of the deck is funicular under $\mathbf{p}_d(x) + \mathbf{K}(x)$, and
- the segment $x \in [k(x_0), k(x_1)]$ of the arch is also funicular under $\mathbf{p}_a(x) - \mathbf{K}(x)$.

As we will see, the hangers transfer forces from one curve to the other according to their needs, which broadens the range of geometries, which allow funicular equilibrium.

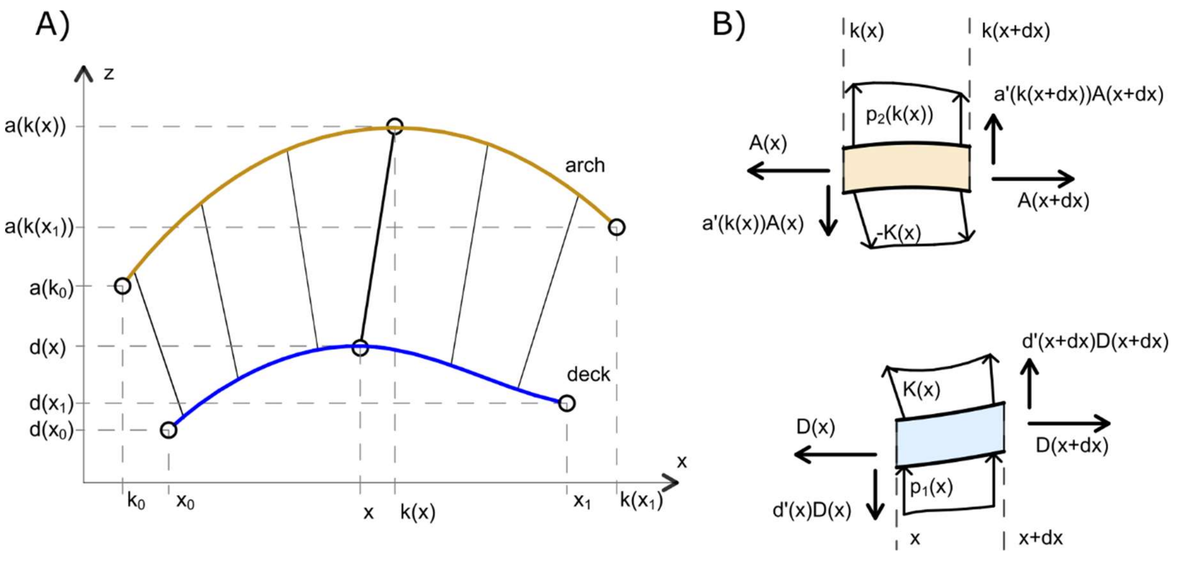

Figure 2: Notation of coupled funiculars (A), and free-body diagrams of infinitesimal pieces of the curves (B). Note, that capital letters represent forces, small letters represent positions or distances.

We emphasize that in practical applications, it is not often convenient to build curves with continuous coupling, however the proposed model is a reasonable approximation of a dense discrete sequence of cables. In an actual design process, the continuous solution can be replaced by a discretized system of coupling cables with arbitrary density.

2.3. Equilibrium equations of coupled funicular curves and a new approach to form-finding

The equilibrium of an infinitesimal segment of a pair of coupled funiculars can be derived in analogy with (1) -(2). It consists of two equilibrium equations of the deck (4) -(5), and two equations of the arch (6) -(7):

$$K(x)\frac{k(x)-x}{|\boldsymbol{a}(k(x))-\boldsymbol{d}(x)|}+p_{dx}(x)+D'(x)=0, \tag{4}$$

$$d''(x)D(x)+d'(x)D'(x)+p_{dz}(x)+K(x)\frac{a(k(x))-r(x)}{|\boldsymbol{a}(k(x))-\boldsymbol{d}(x)|}=0. \tag{5}$$

$$A'(x_d)-K(x_d)\frac{k(x)-x}{|\boldsymbol{a}(k(x))-\boldsymbol{d}(x)|}+p_{ax}\big(k(x)\big)k'(x)=0 \tag{6}$$

$$a''\big(k(x)\big)k'(x)\,A(x)+a'\big(k(x)\big)\,A'(x)-K(x)\frac{a(k(x))-r(x)}{|\boldsymbol{a}(k(x))-\boldsymbol{d}(x)|}+p_{az}\big(k(x)\big)k'(x)=0. \tag{7}$$

Here $A(x)$, and $D(x)$ are the horizontal components of the (unknown) internal normal forces of the two curves (arch, deck) in analogy with $C(x)$ in (1) -(2). Equations (4) -(7) form a coupled system of four equations. By choosing (four) unknown functions and by prescribing all other functions, different types of design problems can be addressed by these equations. The mathematical character of the problem also depends on this choice. A common approach in structural form finding is to prescribe loads, the geometry of one curve and the directions of hangers, and to seek the shape of the second curve along with the auxiliary functions $A(x), D(x), K(x)$. The practical value of this approach lies in that the designer can prescribe the shape of one curve (e.g. main beam of a post-tensioned roof, or deck of a bridge) based on visual or functional requirements, and the shape of the other main component (post-tensioning or bridge arch) is dictated by the laws of physics.

Meanwhile, by using equations (4-7) it is possible to formulate a novel kind of form finding method, which we call *free-form coupling*. In this case, the shapes of *both curves* are prescribed, thus unknown functions are the internal forces in all three members $K(x), A(x), D(x)$ and the function $k(x)$ representing directions of hangers. The main advantage of the free-form coupling approach is that it offers the designer freedom to shape the two visually dominant elements of coupled curves freely, while maintaining the advantageous properties of funiculars. For a brief comparison of the two form finding methods see Table 1.

Table 1: Comparison of two form finding methods.

Method	Prescribed functions	Unknown functions
Traditional form finding	$d(x), k(x), \mathbf{p}_a(x), \mathbf{p}_d(x)$	$a(x), D(x), A(x), K(x)$
Free-form coupling	$d(x), a(x), \mathbf{p}_a(x), \mathbf{p}_d(x)$	$k(x), D(x), A(x), K(x)$

3. Numerical solution of the free-form coupling problem

First, $K(x)$ is expressed in explicit form using (4) -(5) as:

$$K(x)=\frac{-d''(x)D(x)+d'(x)p_{dx}(x)-p_{dz}(x)}{a(k(x))-d(x)-d'(x)(k(x)-x)}\big|\mathbf{a}\big(k(x)\big)-\mathbf{d}(x)\big|. \tag{8}$$

Equation (8) enables us to eliminate $K(x)$ from (5) -(7), to obtain a system of three nonlinear, first order differential equation for $k(x), A(x), D(x)$. With appropriate initial conditions at an arbitrary point $x = x_0$, we obtain a standard initial value problem (IVP):

$$\begin{cases} k'(x) = K(x)\dfrac{a(k(x)) - d(x) - a'(k(x))(k(x) - x)}{a''(k(x))A(x) + p_{az}(k(x)) - a'(k(x))p_{ax}(k(x))}\dfrac{1}{|\mathbf{a}(k(x)) - \mathbf{d}(x)|} \\ D'(x) = -K(x)\dfrac{k(x) - x}{|\boldsymbol{a}(k(x)) - \boldsymbol{d}(x)|} - p_{dx}(x) \\ A'(x) = K(x)\dfrac{k(x) - x}{|\boldsymbol{a}(k(x)) - \boldsymbol{d}(x)|} - p_{ax}(k(x))k'(x) \\ k(x_0) = k_0 \\ A(x_0) = A_0 \\ D(x_0) = D_0 \end{cases} \tag{9}$$

Numerical solutions of this problem were computed with the aid of the ode45 IVP solver in MATLAB environment. Figure 3 shows three examples, in each of which we chose the following shape functions and loads: $a(x) = -0.2\,|x - 5|^{1.7} + 5$ with no load on the arch $\boldsymbol{p}_a \equiv \underline{0}$, and $d(x) = (18^2 - (\mathrm{x} - 5)^2)^{0.5} - 16$ with constant load $p_{dz} = -3$ and $p_{dx} = 0$. The left endpoint was specified as $x_0 = 0$ and the solver was stopped if either x or $k(x)$ reached the terminal value of $x_1 = 10$. The three solutions were obtained by choosing different initial conditions (see figure caption). Comparison of the three solutions reveals that the initial values D_0, A_0 have large influence on the emerging $k(x)$ function, i.e. on how the two curves are coupled.

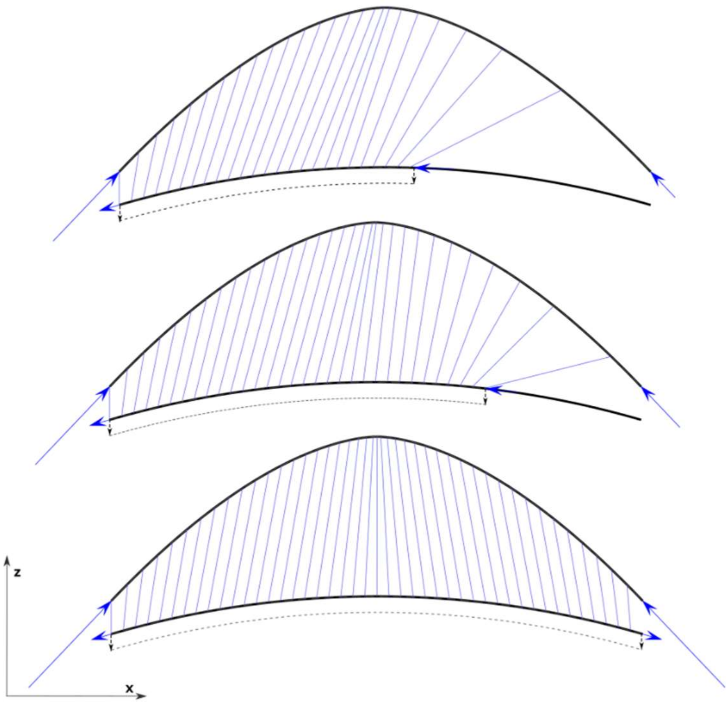

Figure 3: Three solutions of the free-form coupling problem with the same prescribed curves and loads (see main text), but with different initial conditions: $[D_0, A_0, k_0] = [2.54, -12.00, 0]$ (A), $[D_0, A_0, k_0] = [2.54, -13.50, 0]$ (B), $[D_0, A_0, k_0] = [2.54, -15.00, 0]$ (C). The two curves are depicted as thick black curves, the coupling function k(x) is illustrated by a dense sequence of thin (blue online) line segments, the external loads by grey, and the reaction forces at the endpoints (balancing terminal values of internal forces) by arrows.

It is also worth mentioning at this point that the numerical algorithms of IVPs allow loads $\mathbf{p}_a(x)$, $\mathbf{p}_d(x)$ to be prescribed function of the (initially unknown) values $D(x)$, $A(x)$, $k(x)$. This feature is highly useful if the load includes the weight of the arch or the deck, which often varies in accordance with the internal forces in the case of optimal design solutions. We will refer to this extension of the form finding method as free-form coupling under *dependent loads*.

4. Finding solutions with prescribed endpoints

The solution of Figure 2C has the special property that $k(10) = 10$, and thus the solution is symmetric. From a designer's point of view, it is often convenient to prescribe all endpoints of the coupled curves to match geometric constraints. Both left endpoints as well as the right endpoint of the deck can be specified easily within the framework of IVPs by appropriate choice of x_0, x_1, and k_0. However, a prescribed value of the remaining endpoint means that one of the initial conditions for $A(x_0), D(x_0)$ needs to be replaced by a constraint of the form $k(x_1) = k_1$, which turns the problem into a nonlinear *boundary value problem* (BVP) instead of an IVP. There are well-established numerical methods to tackle nonlinear BVPs, such as the shooting method or iterative solution techniques based on finite element method or finite difference method [7], however all of them are computationally much more expensive than the numerical solution of an IVP.

The use of a BVP solver is unavoidable in the case of *dependent loads*. However, a simple workaround is possible if loads are not dependent, i.e. their numeric values are known in advance. In this special case, an adapted version of the IVP is available. Consider the global equilibrium of the whole structure with prescribed endpoints at both ends, under (known) loads, and four (unknown) reaction forces at the endpoints of the curves (Figure 4). The equilibrium of this force system is characterized by 3 independent equations in 4 unknowns, which has in the generic case a one-parameter set of solutions. Thus, it is possible to find many initial values A_0, D_0, which are consistent with global equilibrium. With initial conditions obtained this way, one can solve the original IVP, to obtain a solution, which usually respects the additional boundary condition. In some exceptional cases, this procedure can still yield false solutions of the BVP, but this phenomenon will not be discussed here in detail.

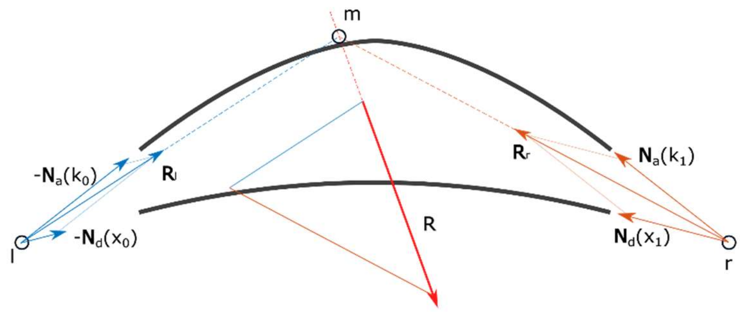

Figure 4: Graphical method of obtaining initial values A_0, D_0, which provide a solution with prescribed endpoints. Thick black curves represent the arch and the deck with prescribed endpoints. **R** is the resultant of all loads, $\mathbf{R}_l, \mathbf{R}_r$ are the resultant of support reactions at the left ($-\mathbf{N}_d(x_0)$ and $-\mathbf{N}_a(k_0)$) and right ($\mathbf{N}_d(x_1)$ and $\mathbf{N}_a(k_1)$) endpoints, respectively.

Instead of writing and solving the equilibrium equations mentioned above, we now present a graphical approach to finding appropriate initial values A_0, D_0 (Figure 4). Here the problem of finding those initial values is reduced to finding the equilibrium of three forces: the resultant of the external loads (**R**), the resultant of the two initial reaction forces ($\mathbf{R}_l$) , and the resultant of the two terminal reactions ($\mathbf{R}_r$).

In the generic case, when the tangents at the endpoints of the arch and the deck are not parallel, the resultant forces $\mathbf{R}_l, \mathbf{R}_r$ can be found as the vector sum of two reactions acting at the intersection points l, r of their lines of action. The three forces $\mathbf{R}, \mathbf{R}_l, \mathbf{R}_r$ are in equilibrium, hence their vector sum equals zero, and their lines of action intersect in a single point m. Accordingly, we can pick an arbitrary point m on the line of action of **R**. Then, the conditions of equilibrium determine $\mathbf{R}_l, \mathbf{R}_r$. Finally, the initial values A_0, R_0 are obtained by the decomposition of $\mathbf{R}_l$ to the sum of two vectors tangential to the two curves. The non-generic case when any of the intersection points l, r does not exist needs minor adaptation of the process outlined above (skipped for the sake of brevity).

The arbitrary position of m reflects, that there is one degree of freedom in picking appropriate initial values. Instead of picking m, it is also possible to prescribe the value of either A_0 or D_0 in advance.

5. Limitations of free-form coupling

In some cases, the IVP (9) does not have a solution, the solution is mechanically irrelevant, or it corresponds to undesirable cable distributions. We assume that the load functions are continuous, and the shape functions have continuous 2nd derivatives. Then two/three types of problem may occur.

1. $k'(x)$ is undefined due to division by 0 in (9),
2. The value of $k'(x)$ dictated by (9) crosses 0,
3. The values of $A'(x)$, and $D'(x)$ given by (9) are undefined due to $\left|\boldsymbol{a}\big(k(x_d)\big) - \boldsymbol{d}(x_d)\right| = 0$.

The first case leads to a singularity in $k(x)$, i.e. it diverges to infinity. The second case lead to solutions, that do not fit the definition of coupled curves, because negative $k'(x)$ changes the topology of the structure. The third scenario happens if two scalar equations are satisfied simultaneously, which is non-generic. This event has not been observed in simulations, and it will not be investigated any further.

For a deeper understanding of the first two scenarios, we rewrite the formula for $k'(x)$, with the aid of (8) and the first equation of (9) as:

$$k'(x) = \left(\frac{a-d-a'(k-x)}{a-d-d'(k-x)}\right) \cdot \left(\frac{-d''D+d'p_{1x}-p_{1z}}{a''A-a'p_{2x}+p_{2z}}\right). \qquad (10)$$

The arguments of the functions have been dropped to shorten notation. The value of $k'(x)$ changes sign if any of the two nominators in (10) is equal to zero:

$$a - d - a'(k - x) = 0, \text{ or} \qquad (11)$$

$$-d''D + d'p_{1x} - p_{1z} = 0. \qquad (12)$$

Equation (11) means that cables become tangential to the arch; thus, cable forces cannot exert a force, which makes the arch funicular (for an example, see Figure 5A)). Equation (12) means that the deck is locally funicular without cables (see (1), (2)), i.e. the cable forces per unit length of the deck drops to zero, whereas the arch usually requires nonzero cable force. Figure 4B shows an example of this scenario.

Similarly, $k'(x)$, is undefined if any of the two denominators in (10) is equal to zero:

$$a - d - d'(k - x) = 0, \text{ or} \qquad (13)$$

$$a''A - a'p_{2x} + p_{2z} = 0. \qquad (14)$$

Equation (13) means that cables are tangential to the deck (Figure 5C)), and (14) means that the arch is locally funicular without cables (Figure 4D).

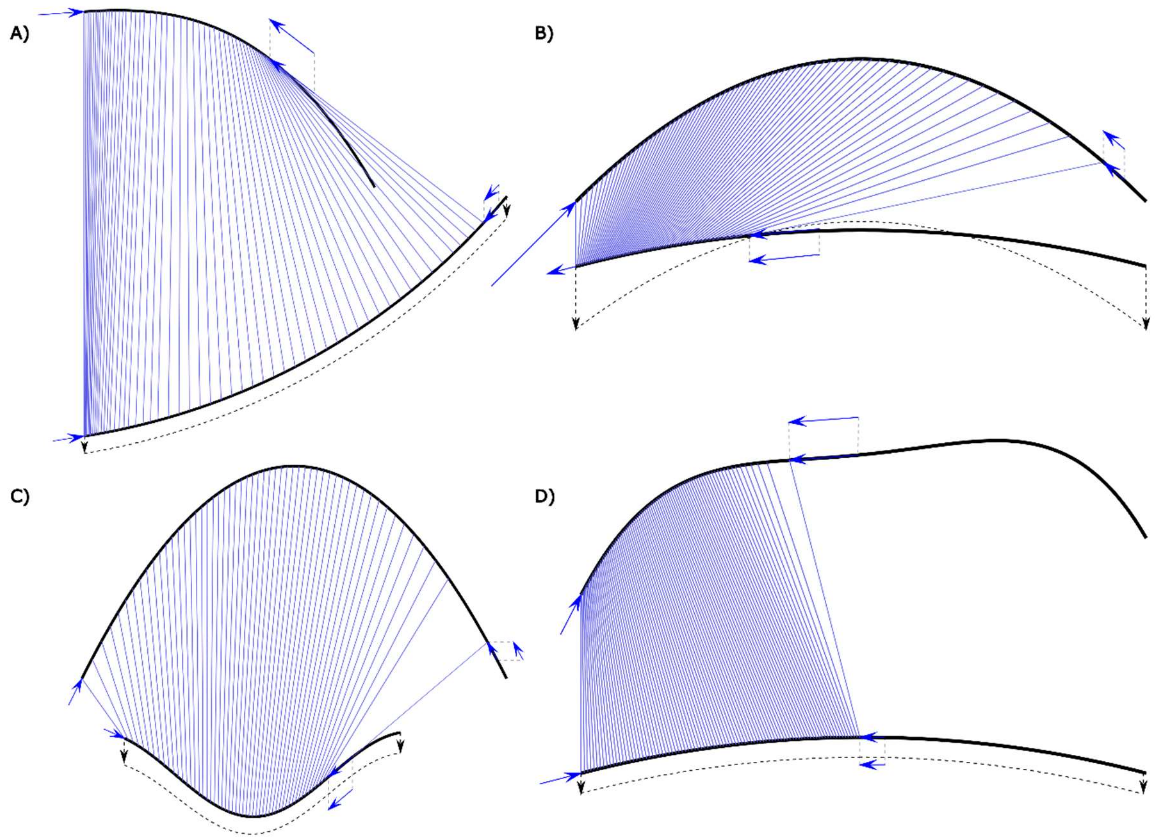

Figure 5: Examples of problematic layouts of coupled curves, which lead to failure of the IVP solver. The arch is unloaded in all cases, and the load of the deck is depicted by dashed lines. The numerical solution was aborted when proximity of any of the failure criteria (eq11) -(eq14) was detected. Blue arrows depict reaction forces ensuring equilibrium at the initial points of the curves and at the terminal points where simulation was aborted. Shifted copies of some arrows are added to enhance visibility. A: Hangers become tangential to the arch, see (11). B: Deck needs no cable force for funicularity as expressed by (12). In this case, failure is caused by vanishing deck load C: Hangers are tangential to deck (13). D: deck needs no cable force for funicularity (14). In this case failure is cause by the inflexion point of the arch.

Summary

This work investigated 'coupled funicular curves', which is a structural topology often used in structural design. The main result is the proposal of a numerical method for finding an appropriate hanger distribution, that makes two main curves of prescribed shapes funicular. This method provides freedom in forming the two main components of coupled curves as dictated by design criteria such as artistic form, and functional or technological demands. An extension of the new free-form coupling method can be used to find cable distributions for prescribed segments of both curves. We showed, that the set of all solutions of the free-form coupling method is a set of internal force functions (in each structural member) and cable position descriptor function with one free parameter. The feasibility of the method was illustrated by an example.

We also proposed a list of failure mechanism of the free-form coupling method. Problems occur, when hangers become tangential to any of the coupled curves, or coupled curves require hanger forces of opposite sense.

These failure types deserve further investigation, since the differential equations may still have solutions albeit with $k'(x) < 0$. Those solutions provide equilibrium in an unconventional way, which may be relevant to structural design. Another area, which deserves further research is free-form coupling of three-dimensional curves, as a potential form-finding tool of spatial arch bridges. Lastly, we recall that the continuous results of the proposed method need to be discretized as part of the post-processing of the solution. Acquiring a continuous solution is highly useful in the sense, that it encapsulates infinitely many discrete solutions. Methods of choosing feasible, and mechanically relevant discretizations (i.e. discrete cable setups) might also be the subject of further research.

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