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Characterizing spatial structures with optimal strength or stiffness

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Abstract

In the context of the current climate emergency, making efficient use of structural materials is vital. Alongside this, structural designs must achieve both the strength and stiffness that are required for the structure to operate with safety and comfort. Topology optimization, e.g. using the ground structure method, can provide a powerful approach to obtaining structural forms which achieve these requirements with minimum material usage. When the structure is designed based on just a single dominant loading case, then an optimal design exists which is simultaneously the strongest and stiffest form possible. However, when the structure must resist a wider range of loading cases, then the stiffest design and the strongest design are no longer the same. Designs of optimal strength have been well studied both numerically and analytically. However, until recently, study of the topologies of optimal stiffness structures has been limited to relatively coarse resolution numerical results. Recent developments by the author have unlocked high-resolution results for planar 2D trusses of optimal stiffness. In this contribution, that approach – the ground structure method, with adaptive member adding – is extended to axially loaded 3D problems of optimal stiffness. It is shown that the forms of optimal stiffness structures are significantly and qualitatively different to optimal strength designs.

Keywords: Optimization, Ground Structure Method, Truss Topology Optimization, Compliance-based optimization, Strength-based optimization

1. Introduction

Structural materials, such as concrete or steel, are highly carbon intensive in their production, transportation, and construction. Thus, is it imperative to make the most efficient use of these materials in structural design. Optimization approaches can be a valuable tool to facilitate this. However, with any optimization-based approach, it is of vital importance to ensure that the mathematical problem posed suitably represents the real-world scenario. One key division within the structural optimization community is between design for optimal strength, and design for optimal stiffness. This distinction is unnecessary for simple academic problems containing just a single load-case; however, the two approaches obtain significantly different designs for structures subjected to a variety of possible loadings, as all real-world structures are. Furthermore, real-world structures typically must fulfil demands relating to both strength and stiffness. Despite this, these two areas of the research community are rather disparate, and there is little work on comparatively evaluating the solutions of optimal strength and optimal stiffness, or on transferring techniques from one field to the other.

This contribution takes the adaptive ground structure (member adding) approach [1], which is common in the strength-based optimization community, and applies it to spatial problems of optimal stiffness design. The first results for stiffness-based problems using this approach can be found in [2] (for planar problems only) along with a more comprehensive review of the background literature.

2. The ground structure method for truss topology (layout) optimization

The ground structure method is an approach for truss topology, or layout, optimization. As shown in Figure 1a, the problem is first defined, including loads, supports and the allowable design domain (note that an initial guess for the structure is *not* required). The domain is then discretised using a large number of nodes (Figure 1b), each pair of which are then connected by a potential member to form the ground structure (Figure 1c). Finally, a convex optimization problem is formulated and solved to give the areas of the elements in the optimal structure (Figure 1d). The problem to be solved differs for strength- or stiffness-based design.

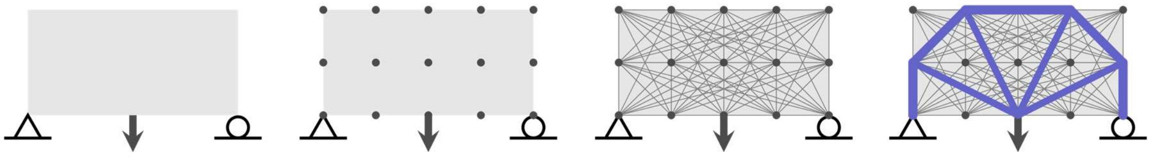


Figure 1: The ground structure method, process stages. (a) Problem definition. (b) discretization of design domain using nodes. (c) connection of all pairs of nodes with potential elements. (d) optimal set of elements selected via convex optimization.

2.1 Strength-based design (rigid-plastic model)

The ground structure method is often used to find the strongest possible design, by applying principles of limit analysis, i.e. with a rigid-plastic material model. This formulation was first suggested by [3] with later developments by [1, 4]. For this strength-based design paradigm, the optimization problem which must be solved to move from Figure 1c to 1d can be stated for a problem containing m potential elements and n nodes as:

$$\min_{\boldsymbol{a},\boldsymbol{q}_k} V = \boldsymbol{l}^T\boldsymbol{a} \qquad \text{minimise volume} \qquad (1a)$$

$$\text{subject to } (\boldsymbol{B}\boldsymbol{q}_k = \boldsymbol{f}_k)_{\forall k} \qquad \text{equilibrium} \qquad (2b)$$

$$(\sigma_Y\, \boldsymbol{a} - \boldsymbol{q}_k \geq \boldsymbol{0})_{\forall k} \qquad \text{yield stress (tension)} \qquad (3c)$$

$$(\sigma_Y\, \boldsymbol{a} + \boldsymbol{q}_k \geq \boldsymbol{0})_{\forall k} \qquad \text{yield stress (compression)} \qquad (4d)$$

Where $\boldsymbol{l} = [l_1, l_2, \dots, l_m]^T$ is a vector containing the lengths of each potential element, $\boldsymbol{a} = [a_1, a_2, \dots, a_m]^T$ is a vector of optimization variables representing the cross-section area of each potential element, which can be used to calculate the total structure volume V. The axial force in each element in a certain load-case k is represented by the optimization variables $\boldsymbol{q}_k = [q_1, q_2, \dots q_m]^T$. The matrix $\boldsymbol{B}$ has dimensions $2n \times m$ for 2D problems or $3n \times m$ for 3D problems and contains direction cosines which may be multiplied by $\boldsymbol{q}_k$ to resolve the axial forces along the global coordinate axes. Then $\boldsymbol{f}_k$ is a vector of length $2n$ for 2D problems or $3n$ for 3D problems, which contains the externally applied loads in case k. A specified plastic yield stress σ_Y must be defined, here this will be assumed to be equal in tension or compression.

2.2 Stiffness-based design (linear-elastic model)

For the problem of stiffness-based (linear-elastic) design, the total structural volume will again be minimised, but now with a constraint on the stiffness. A range of formulations for this and equivalent problems (e.g. maximising stiffness for fixed material usage) can be found in [5].

Here, to quantify stiffness, a limit W_k will be set on the work done by the applied loads (i.e. the applied force multiplied by displacement, summed over every degree of freedom at each node). This limit is applied in each load-case individually. To transform this limit to a setting comparable to the static, force-based problem in (1) requires some algebraic manipulation, which is outlined in the following paragraph.

First, it must be noted that the external work in a load-case will always be equal to the internal work, allowing the constraint to be re-stated as "total *internal* work in load-case k $\leq W_k$". Internal work can be calculated for each element, and then summed to give the total for the load-case. For this purpose, additional optimization variables, $p_{i,k}$, represent the internal work of element i in load-case k. Then, for a single element, p is defined to be $\frac{1}{2}Kx^2$, where K is the stiffness of the element and x is the extension of the element. By rearranging this, and using standard linear-elastic relationships (e.g. elastic modulus $E = \frac{\sigma}{\epsilon} = \frac{\text{stress}}{\text{strain}}$) it is possible to obtain: $p = \frac{1}{2}Kx^2 = \frac{1}{2}\frac{q}{x}x^2 = \frac{1}{2}ql\epsilon = \frac{1}{2}ql\frac{\sigma}{E} = \frac{1}{2}ql\frac{q/a}{E} = \frac{1}{2}\frac{l}{E}\frac{q^2}{a}$.

This is then combined with the objective function and equilibrium constraints as before:

$$\min_{\boldsymbol{a},\boldsymbol{q}_k,\boldsymbol{p}_k} V = \boldsymbol{l}^T\boldsymbol{a} \qquad \text{minimise volume} \qquad (2a)$$

$$\text{subject to } (\boldsymbol{B}\boldsymbol{q}_k = \boldsymbol{f}_k)_{\forall k} \qquad \text{equilibrium} \qquad (2b)$$

$$\left(\frac{1}{2}\frac{l_i}{E_i}\frac{q_{i,k}^2}{a_i} \leq p_{i,k}\right)_{\forall k} \qquad \text{internal work, per-element} \qquad (2c)$$

$$\left(\sum_{i\in M} p_{i,k} \leq W_k\right)_{\forall k} \qquad \text{limit internal work, total} \qquad (2d)$$

$$\boldsymbol{a} \geq \boldsymbol{0} \qquad (2e)$$

Whilst this formulation is no longer linear, it can be posed as a convex conic problem, which can be solved at a similar speed to a linear problem when using modern solvers. For a single load-case, there is a statically determinate solution, which is the optimal design for both (2) and (1); i.e. with equal values of $\boldsymbol{q}_k$. The appropriate cross-section areas must be calculated according to the relevant material and problem parameters for each problem, i.e. σ_Y for plastic problems solving (1) or W_k, E for the elastic problem solving (2).

2.3 Computational considerations

Both (1) and (2) are convex optimization problems, and thus can be easily solved to global optimality, even with large numbers of variables. However, for very dense nodal grids, the computation time may still become significant, whilst the memory requirements may outstrip the capabilities of typical laptop or desktop computers. In such cases, the adaptive ground structure method can be used.

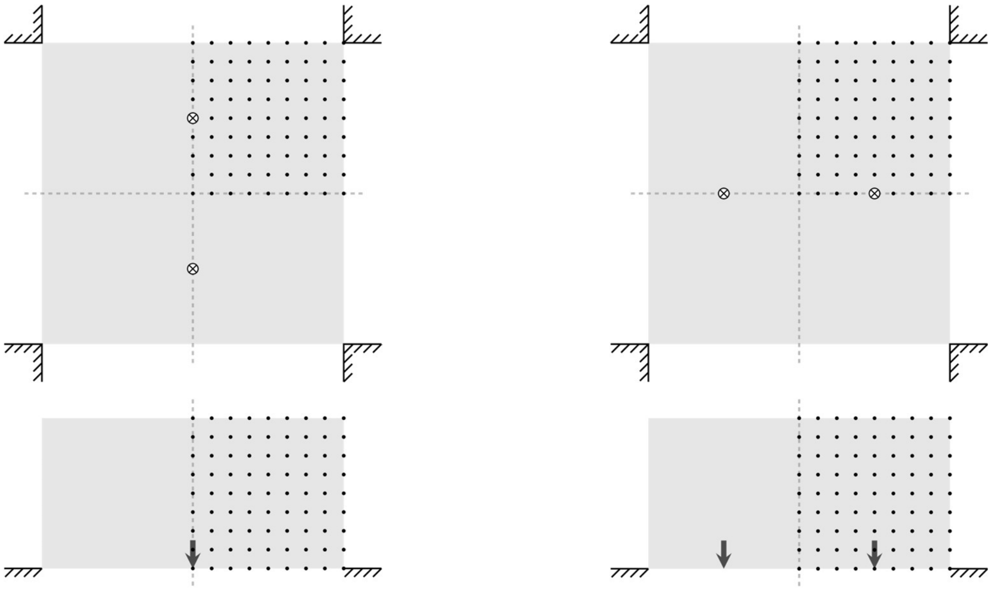
A full derivation of this approach is omitted for brevity, and interested readers are referred to [1] or [2] for descriptions within the settings of strength-based or stiffness-based design respectively. Key points are that the adaptive approach allows the problem to be solved using a reduced ground structure (e.g. adjacent connectivity), which is iteratively increased. It can be mathematically proven that the result obtained through the adaptive approach will be identical to that obtained by solving the full problem (again, for a full proof of this, readers are referred to the aforementioned works), however the computational requirements, in terms of both time and memory, can be reduced by orders of magnitude.

3. Structures of optimal strength or optimal stiffness: Numerical results.

Mathematically, extension of the ground structure methods to 3D problems poses few challenges, as noted in Section 2. The principal difficulty is in managing the geometric input and output in 3D space. For this reason, the Python code from [2] has been adapted to function within the Grasshopper parametric modelling environment of Rhino 8. Numerical volumes will depend on additional parameters e.g. deflection limits, so are not comparable between elastic and plastic results.

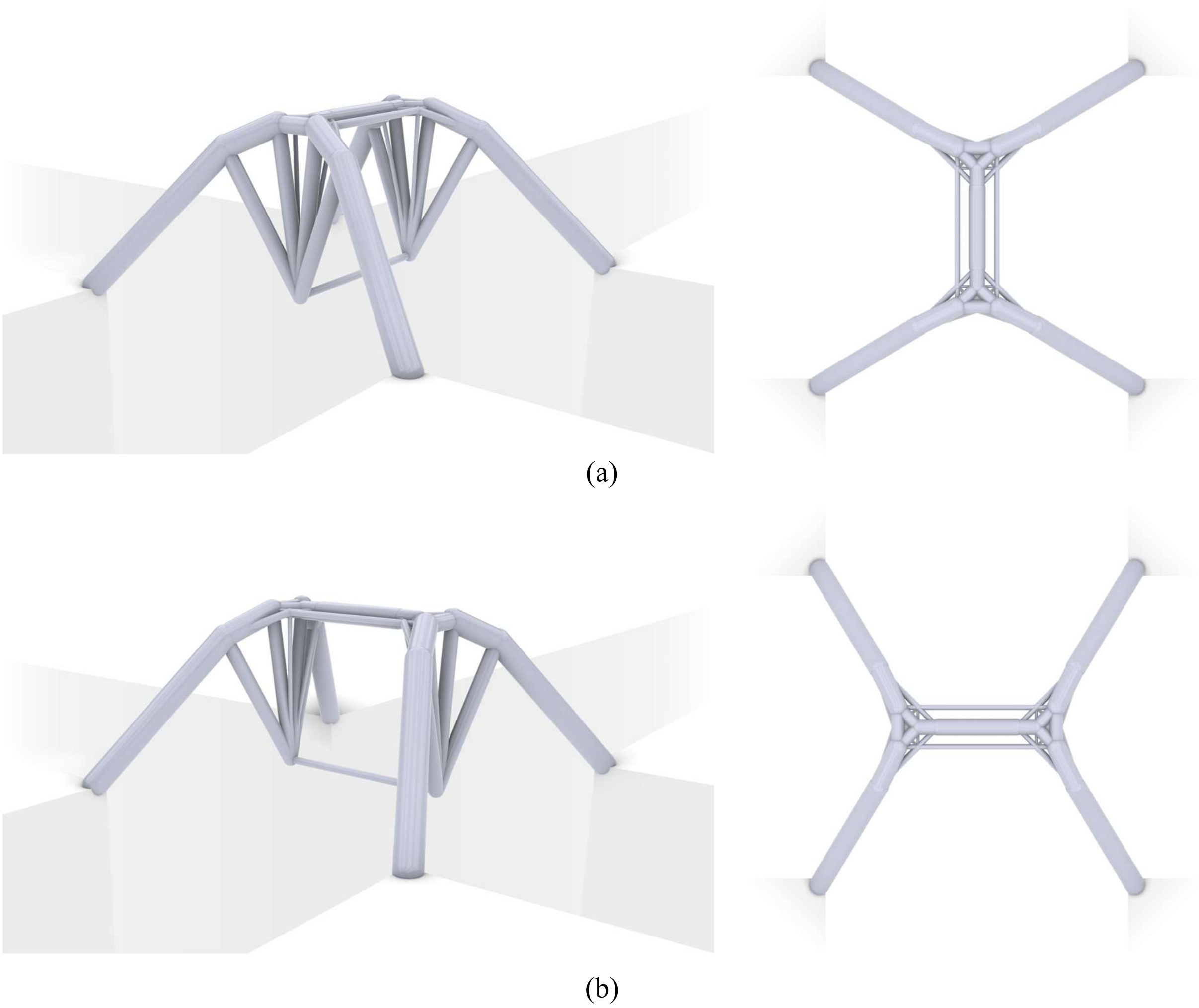
3.1. 3D Canopy Example

The approach will be tested on a 3D problem, as specified in Figure 2. The design domain is a cuboid with equal width and depth L and a height $L/2$. Pin supports are provided in the bottom 4 corners of the domain, and loads are also applied at the base elevation as shown. The loaded points are halfway between the midpoint of a side, and the centre of the domain, and pairs of opposite loadings are applied simultaneously, leading to a problem with 2 load-cases. By symmetry, it is possible to explicitly model just a quarter of the domain, which has been discretized using nodes at a spacing of $\frac{L}{16}$, leading to a ground structure containing 220,280 potential elements.



(a) (b)

Figure 2: 3D Example: Problem specification. (a) Load-case 1 and (b) Load-case 2. Elevation and plan shown for each case. Note that ⊗ is used in plan to indicate loading location (downwards, at base of domain).



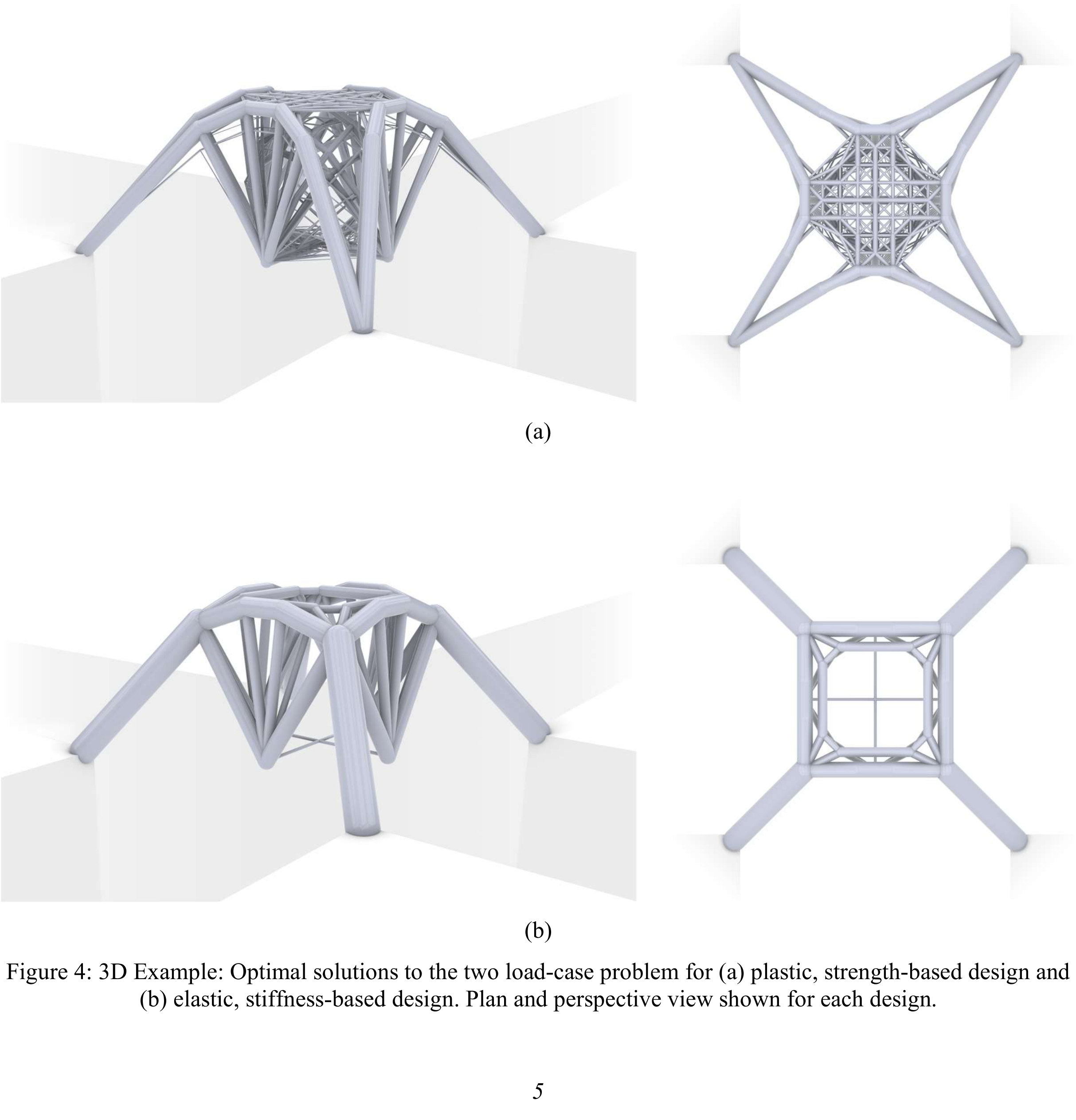
(a)

(b)

Figure 3: 3D Example: Optimal structures to carry each load-case independently. Plan and perspective views.

For comparison purposes, structures optimized for just one of the two load-cases are shown in Figure 3. As these structures are single load-case solutions, they are statically determinate and thus valid for any material model, i.e. for both strength-based and stiffness-based design. One naïve approach to multiple load-case design suggests a strategy of combining these two structures to create a design capable of resisting either loading. For strength-based design, this gives a feasible solution, but one that uses much more material than is necessary – over 46% more for this example. For stiffness-based design, this approach is typically not possible, as the required displacements would not be compatible. (E.g. here, the elements across the centre of the domain would need to displace separately according to which case was loaded.) Even assuming that detailing could somehow be provided to facilitate this, the naïve approach still requires 41% more material than necessary in this case.

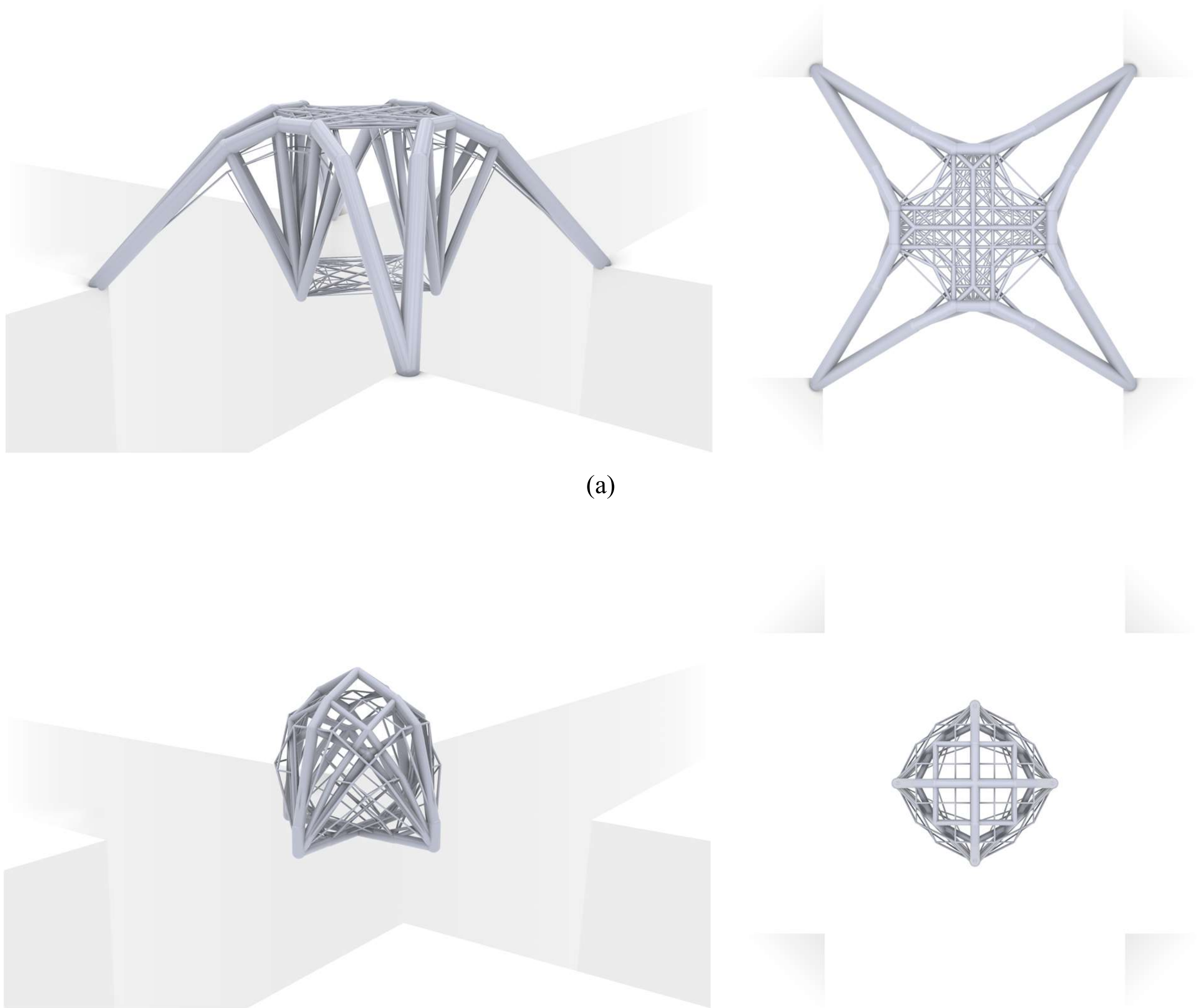
Instead, correct solutions should be obtained by solving the multiple load-case formulations (1) or (2). The solutions to these are shown in Figure 4. The optimal strength and stiffness designs are now significantly different. For example, the optimal strength design contains a dense web of elements in the centre of the domain, whilst this region of the optimal stiffness design is relatively empty, with just a single cross of elements on the base of the domain. Another difference is that the 'legs' which connect to the support points go directly towards a point almost above the load in the strength-based design, while in the elastic design they are at 45^{o} to the domain edges, with a distinct split to go perpendicularly and form a square above the loads.



(a)

(b)

Figure 4: 3D Example: Optimal solutions to the two load-case problem for (a) plastic, strength-based design and (b) elastic, stiffness-based design. Plan and perspective view shown for each design.



(a)

(b)

Figure 5: 3D Example: Decomposition of optimal plastic design into component load-cases. (a) Half of load-case 1 + load-case 2. (b) Half of load-case 1 - load-case 2. Note that the loading in part b can achieve equilibrium without requiring additional reaction forces, therefore this structure does not connect to the supports.

Both solutions contain features which are commonly observed in optimal truss structures for single load-cases. Particularly evident in this case are the fans of tensile elements from the loads and accompanying arches of compression elements.

The observation of these characteristics in the stiffness-based results cannot be easily explained by any known principle. Nonetheless, this suggests that stiffness-based solutions are in some way linked to single load-case designs and that further research in this avenue may be fruitful.

For the strength-based design, it is not surprising that such features are observed; this is because the Prager-Nagtegaal superposition principle [6] can be used to decompose the multiple load-case result in Figure 4a into component structures, obtained through single load-case optimization results. These single load-case results are *not*, however, those shown in Figure 3. Instead, these component load-cases are defined by taking the half of the sum or difference of the two applied load-cases. The single load-case results for these component load-cases are shown in Figure 5. By comparing this with Figure 4a, the more complex multiple load-case result can be more easily interpreted. It can also be clearly seen that the optimal stiffness design (Figure 4b) is not closely related to these component structures.

3.2 2D Cantilever Example

To better illustrate the superposition principle for strength-based design, and further illustrate the differences with optimal stiffness designs, a simpler problem will be employed. The problem consists of a cantilever, with permitted height equal to half the length. Point loads are applied at the tip of the cantilever, and may be applied at either the top or bottom of the domain as shown in Figure 6a/b. The optimal strength structure, obtained by solving (1), is shown in Figure 6c. Again, this structure displays forms characteristic of optimal structures, particularly the two near-orthogonal families of intersecting tension and compression elements. The component structures for this case consist of a 'difference' case, where the optimal structure is simply a vertical line on the left of the domain, connecting the two points of load application. Combined with this is the 'sum' case, for which the optimal design comprises the remaining elements in Figure 6c.

Figure 6e annotates the optimal strength result with load-paths to give a conceptual explanation of the superposition principle (considering the case when the loading is applied to the top of the domain). It may be imagined that the applied load is split into two equal portions; the first of which is carried directly through the 'sum' structure to the supports; this is indicated with blue arrows in Figure 2e. Meanwhile, the other portion is transferred along the vertical element of the 'difference' component structure to the location of the other loaded point. From there, it again enters the 'sum' component of the structure and is transmitted to the support. This highlights why this superposition principle cannot be directly applied to stiffness-based design. The load-path indicated in black is longer, and therefore less stiff, than the load-path indicated in blue, and so they cannot be equally utilized in a linear-elastic framework.

Meanwhile, the optimal stiffness result, solving (2), is shown in Figure 6d. Again, there are forms characteristic of single load-case optimal designs, including the near-orthogonal tension and compression elements. However, it can again be seen that this optimal stiffness solution is not generated by the component solutions previously mentioned. Nonetheless, there are single load-case designs which do present similarities to the structure in Figure 6d. Specifically, single load-case structures for point-loads at other locations along the right edge of the domain can be distinguished, such as the examples shown in Figure 6f. However, further work is required to establish how this observation may be generalised to other cases, as there is not a clear analogy to this in the example shown in Section 3.1.

4. Concluding Remarks

This contribution has demonstrated that the adaptive ground structure method may be successfully applied to the problem of optimal stiffness design in both 2D and 3D cases. Consideration of the full multiple load-case problem was shown to be equally important for optimal stiffness design as it is for optimal strength design, allowing over a 40% material saving over a naïve design approach based on identifying the optimal design for each case separately.

The optimal stiffness designs were observed to contain many forms typical of statically determinate, single load-case optimal results, including fan regions and regions of near-orthogonal tension and compression elements. Furthermore, there was some indication of superposition-like characteristics. However, the optimal stiffness solutions were notably different to the optimal strength designs, which could be interpreted by use of the Prager-Nagtegaal superposition principle. This indicates that further investigations are required in this area to fully establish a means of interpreting optimal stiffness designs, and that the adaptive ground structure approach can be a valuable numerical tool for informing such investigations. The method is fully generalizable to any configuration of loads and supports.

Finally, the adaptive ground structure approach for optimal stiffness design has been shown to be conveniently implemented within a commercial parametric modelling package (Rhino/Grasshopper), and further work is planned to make this methodology more widely available on such platforms.

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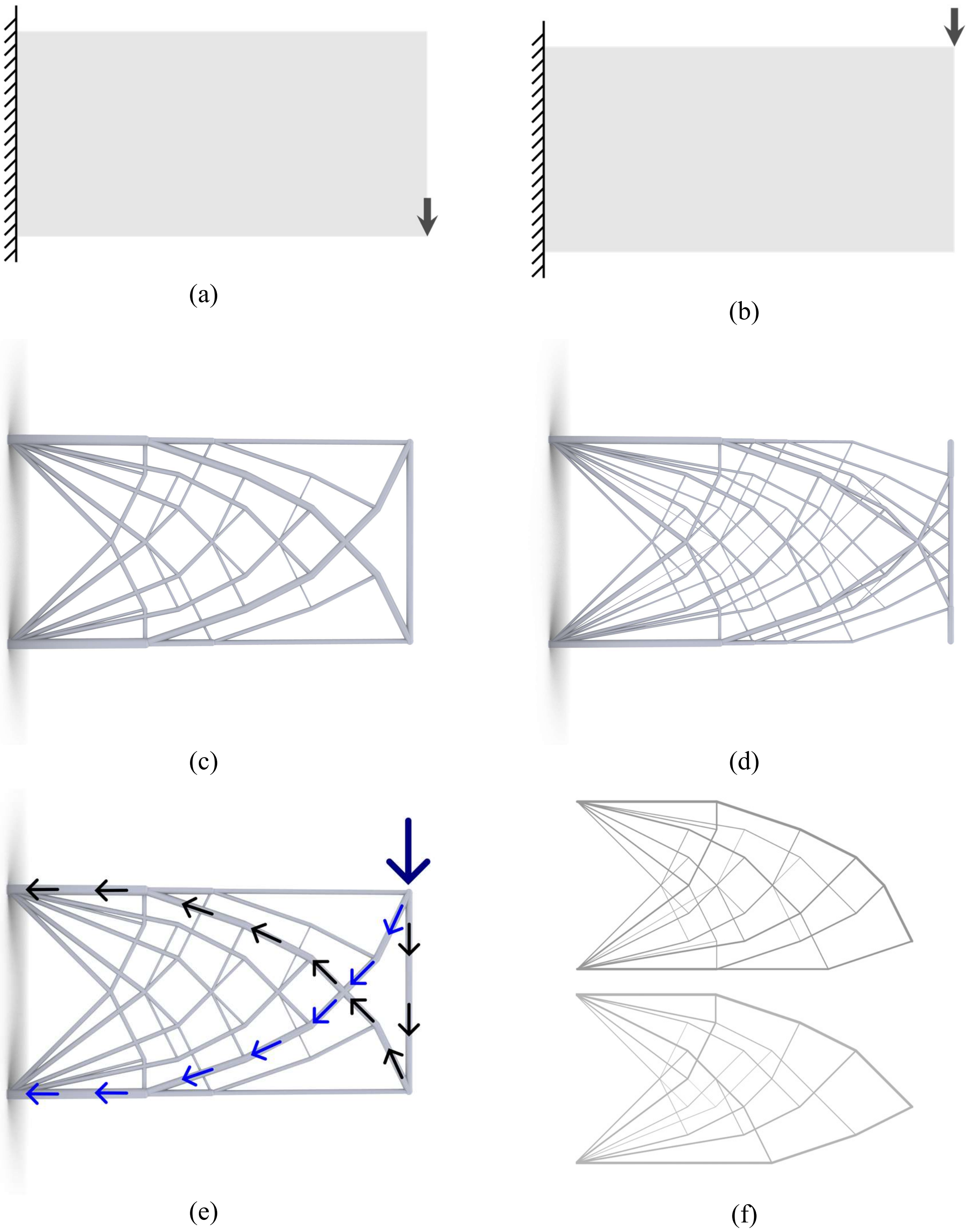

Figure 6: 2D Example: (a) and (b) Problem specification under load-cases 1 and 2 respectively. (c) Solution for optimal strength (d) Solution for optimal stiffness. (e) Annotated solution for optimal strength, illustrating the superposition principle. (f) Some single load-case solutions with different loading positions.

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