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On structural morphology of snow vaults

Esko JÄRVENPÄÄ*, Antti H. NIEMI[a],

*WSP Finland
Koskitie 43 C109, 90500 Oulu, Finland
eojarvenpaa@gmail.com

[a]Civil Engineering Research Unit, Faculty of Technology, University of Oulu
Pentti Kaiteran katu 1, 90570 Oulu, Finland

Abstract

The article discusses the application of structural morphology to snow vaults, with the traditional forms being the parabola, the catenary and the circle. Additionally, the less commonly known constant stress arch is introduced. The parabolic stand-alone momentless arch requires a constant vertical load throughout the entire span, with decreasing thickness from crown to base. In contrast, the catenary arch maintains a constant thickness over the entire arch. The shape function is of the hyperbolic cosine type. The article suggests that the circular shape is only a reasonable structural shape for low rise ratios. In a constant stress arch, the thickness increases from crown to base, and the shape follows a logarithmic cosine function. The article compares snow vaults and finds that the constant stress shape is superior to the catenary and parabola shapes. The constant stress vault allows for considerably larger span lengths compared to parabolic or catenary vaults. This vault type has not yet been used in the design of snow structures. The constant stress vault, characterized by its low compressive stress, allows for the use of weaker and more affordable materials such as poured adobe and adobe bricks in construction. To ensure safety, it is necessary to control and recalculate the load bearing capacity of snow vault constructions due to the potential for large deformations during their service life.

Keywords: snow constructions, arch, vault, constant stress vault, weak material

1. Introduction

The thrust line theory is a well-known design principle for stone and concrete arches. It assumes that compressive strength is not critical to the load-bearing capacity in stone structures (Gerhardt and Pichler [1]). To ensure a cross-section without tensile stress in the arch, the thrust line must be located in the middle third of the arch rib cross-section. However, this assumption is not applicable to snow vaults.

The traditional form of snow structure used in Greenland and Alaska is the igloo (Handy [2]). A practical design guideline for snow structure design has been published in Finland (RIL [3]). The guideline requires static calculations for snow structures, with simplified instructions provided for structures that have not undergone separate structural calculations. The Lapland University of Applied Sciences has released a practical guide for designing snow and ice structures based on field tests. The guide is intended for designers and authorities responsible for implementing snow structures (Ryynänen [4]). These publications do not have a code status, but they are often applied in practice.

When designing and constructing snow and ice vaults, it is crucial to ensure that the thrust line closely follows the axis of gravity. The moment-free form can only be achieved for permanent loads, as all other variable loads impose bending stresses. In snow structures, the weight of the snow is the primary load, so the focus should be on designing the shape.

Research has shown that the most efficient shape for material flow when supporting an evenly distributed vertical load between two support points is the parabolic shape (Tyas [5]).

In 1675, Robert Hooke first published the principle of the catenary arch (Hooke [6]). Although he recognized that the arch was not a parabola, he was unable to determine its mathematical solution (Bukowski [7]). The equation for the shape of the arch was published in 1691, after the development of differential and integral calculus (Heyman [8], Alassi [9]).

The arch can be designed to maintain a constant compressive stress throughout the structure (Marano et al. [10] and [11]). This arch shape has only been studied in recent years (Lewis [12]). The constant stress arch leads to a low compressive stress, which is a significant advantage for snow structures, allowing much longer spans than parabola or catenary shapes.

The circular shape is effective in handling compressive loads but is less suitable for vertical loads. To construct a stand-alone, momentless circular vault, it is necessary to increase the thickness of the vault towards the base as its height increases. The circular arch shape is only viable for relatively shallow arches.

The shape of the momentless arch does not precisely follow the shape along the centroidal axis. The transfer of the center of gravity above the centroidal axis is affected by the thickness and curvature of the arch rib (Nikolic [13]). This article will not discuss this topic further.

When building a snow vault, it is important to note that the shape of the inner soffit differs from the shape along the axis of gravity. The construction drawings should provide the shape of both the inner and outer soffits.

In cold regions, such as popular tourist destinations, snow and ice are often used as temporary building materials. One of the largest tourist events in the world is the annual snow sculpture festival in Harbin, China. Similarly, in North Finland, snow structures are built every year. Figure 1 displays the interior of a snow castle in Kemi, Finland.

Figure 1: Snow castle in Kemi, Finland, year 2019

2. Vault forms

2.1. Parabolic snow vault

The equation for the parabolic arch shape in the coordinate system shown in Figure 2 is

$$y = -\frac{4h}{l^2}x^2 + h\,, \qquad -l/2 < x < l/2. \tag{1}$$

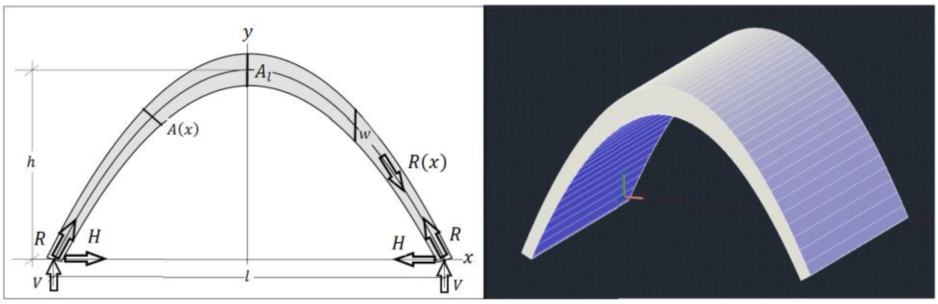

Figure 2: Parabolic stand-alone momentless snow vault

The horizontal force *H* acting on the vault, corresponding to the vertical load *w* is then given by

$$H = \frac{wl^2}{8h}, \tag{2}$$

and the thrust force of the vault at point *x* is

$$R(x) = H\sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{3}$$

Consequently, the normal force *R* at the base is

$$R = \frac{wl^2}{8h}\sqrt{1 + \frac{16h^2}{l^2}}. \tag{4}$$

If we denote the area of the cross-section at the vertex by the symbol A_l, the area of the cross-section *A(x)* at the point *x* is

$$A(x) = \frac{A_l}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}. \tag{5}$$

The dead load of the stand-alone vault is $w = qA_l$, where q is the unit weight of the snow. Combining Equations (4) and (5) the compressive σ at the base is

$$\sigma = q\left(\frac{l^2}{8h} + 2h\right). \tag{6}$$

The compressive stress reaches its minimum when

$$\frac{l}{h} = 4. \tag{7}$$

The maximum span length l_{pmax} of the momentless stand-alone parabolic arch is given by the rise ratio 4. Using Equations (6) and (7) the span length can be calculated as

$$l_{pmax} = \frac{\sigma}{q}. \tag{8}$$

2.2.1 Stress in stand-alone momentless parabolic arch

The compressive stress at the base of a momentless parabolic arch is determined by the rise ratio, unit weight, and span length. This stress can be calculated using the formula

$$\sigma = k_p q l, \tag{9}$$

where the coefficient k_p is determined in relation to the rise ration by using Equation (6). The calculated results of the coefficient k_p are shown graphically later in Figure 5.

2.2. Catenary snow vault

The equation for the catenary shape arch in the coordinates shown in Figure 3 can be written as follows:

$$y = -a \cosh\left(\frac{x}{a}\right) + a + h, \tag{10}$$

where $a = H/w$, H is the horizontal force and w is the weight per unit length.

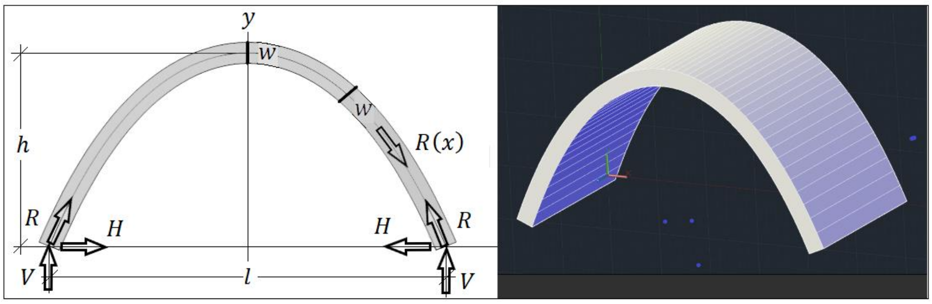

Figure 3: Catenary stand-alone snow vault

Corresponding to the parabola above, the thrust force $R(x)$ at point x is

$$R(x) = H\sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{11}$$

By applying Equation (10) to Equation (12) the expression of the thrust $R(x)$ is

$$R(x) = H\sqrt{1 + \left(\sinh\left(\frac{x}{a}\right)\right)^2} = H\cosh\left(\frac{x}{a}\right). \tag{12}$$

The horizontal force H is directly proportional to the arch length with the same rise ratio. The expression for the thrust force R at the base can therefore be written as:

$$R = aw * \cosh\left(\frac{l}{2a}\right). \tag{13}$$

To determine the minimum value of the normal force $R(x)$ at the base, we consider the unit vault with span length $l = 1$ and cross-section weight per unit length $w = 1$ and differentiate Equation (12). In this case, the horizontal force $H_0 = a_0$ at $x = ½$, and

$$\frac{d\big(R(a_0)\big)}{da_0} = \cosh\left(\frac{1}{2a_0}\right) - \frac{\sinh\left(\frac{1}{2a_0}\right)}{2a_0} = 0, \tag{14}$$

which is minimized when $a_0 = 0.416778$.

Since $y = 0$ at $x = ½$ in Equation (10), the height of the unit length vault can be calculated as $h_0 = 0.337662$. Therefore, the rise ratio that achieves the minimum normal force R at the base is

$$\frac{l}{h_0} = \frac{1}{0.337662} = 2.961. \tag{15}$$

This rise ratio also determines the maximum span length l_{cmax} of the catenary vault when the compression stress σ at the base is the criteria. The formula for this is

$$l_{cmax} = \frac{\sigma}{0.416778 * q\cosh\left(\frac{1}{2 * 0.416778}\right)} = 1{,}325 * \frac{\sigma}{q}, \tag{16}$$

where q is the unit weight of the snow.

2.2.1 Stress in catenary stand-alone arch

The compressive stress of the catenary stand-alone arch at the base depends on the rise ratio, unit weight, and span length. This stress σ can be calculated using the following formula:

$$\sigma = k_c q l. \tag{17}$$

Equation (10) is used to solve for the corresponding parameter a and rise relation using a unit vault, while Equation (12) is used for axial force calculation. The coefficient k_c is calculated and presented graphically later in Figure 5.

2.3. Circular snow vault

The axis of gravity of the circular shape arch in the coordinates shown in Figure 4, can be expressed as a function of its span length and height as follows:

$$y = \left(\frac{1}{8h}\right)\left(4h^2 - l^2 + \sqrt{-64x^2h^2 + 16h^4 + 8h^2l^2 + l^4}\right). \tag{18}$$

Figure 4 illustrates how the thickness of the moment-less circular stand-alone arch increases towards the base. The stress in the arch decreases from the top to the base (Williams [14]). There is a limit to the rise-to-span ratio of a circular vault where the soffits (the inner surfaces) intersect themselves. Typically, this limit is about 2.3 depending on the snow density.

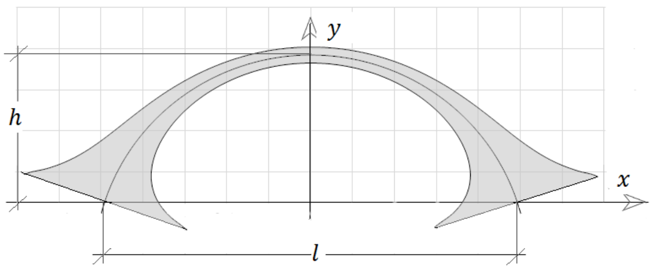

Figure 4: Example of a stand-alone momentless circular snow vault.

2.4. Constant stress snow vault

Based on the coordinates in Figure 5, Equation for the constant stress shape arch is

$$y = \frac{1}{b}\ln(\cos(bx)) + h, \tag{19}$$

where $b = q/|\sigma|$. The symbol q is the unit weight of the material and σ is the stress.

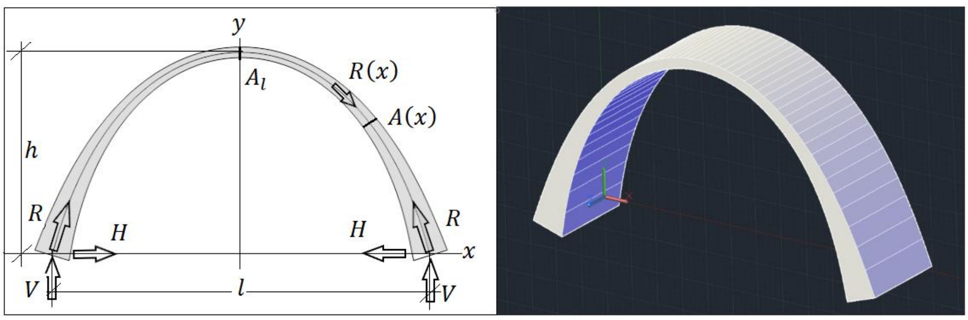

Figure 5: Constant stress vault

The thrust *R(x)* can be calculated now as

$$R(x) = H\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = H\sqrt{1 + (\tan(bx))^2}. \tag{20}$$

By choosing the cross-section A_l at the apex, the horizontal force H is

$$H = A_l\sigma. \tag{21}$$

The cross-sectional area at point x can be calculated as follows (Williams [14]):

$$A(x) = A_l e^{-\ln(\cos(bx))}. \tag{22}$$

The thrust R, at the base is then:

$$R = A_l\sigma\sqrt{1 + (\tan(bx))^2}. \tag{23}$$

Using the formula $R^2 = V^2 + H^2$ and Equations (21) and (23), the vertical support reaction V_y at the base can be expressed as:

$$V_y = A_l\sigma \tan\left(\frac{bl}{2}\right). \tag{24}$$

2.4.1 Stress in constant stress stand-alone arch

The compressive stress of a constant stress arch is determined by the rise ratio, unit weight and span length. The stress σ can be expressed as:

$$\sigma = k_{cs}ql. \tag{25}$$

To solve the coefficient k_{cs}, Equation (19) is used. First, the coefficient b is solved using a unit arch, span length l = 1. The stress is then solved for different rise ratios. The calculated values of k_{cs} are graphically shown in Figure 6.

2.4.2 Ultimate span length of constant stress arch

To achieve the maximum span length of a constant stress arch, the arch's height must be increased infinitely, resulting in a rise ratio of zero. The ultimate span length based on compressive stress σ can be calculated using the following equation (Marano et al. [11]):

$$l_{csult} = \frac{\pi\sigma}{q}. \tag{26}$$

3. Comparison of stresses in parabolic, catenary and constant stress vaults

In the previous sections, we introduced the stress calculation using the coefficients k_p, k_c, and k_{cs}. Figure 6 illustrates stress magnitudes in stand-alone momentless vaults, as shown in the same figure.

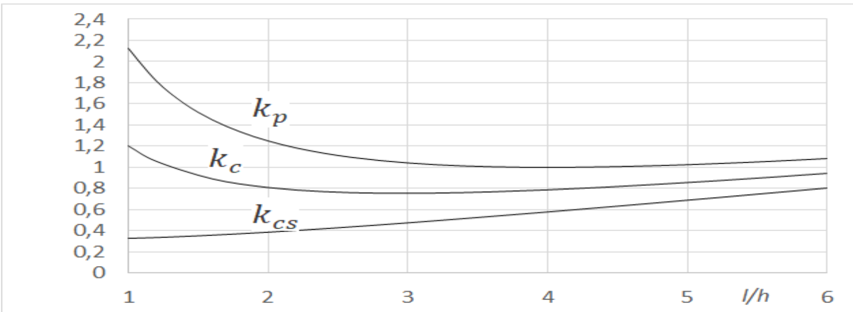

Figure 6: Stress coefficients of parabola, catenary and constants stress vault in stand-alone vaults

Figure 6 shows that as the height of the vaults increases, the stresses become increasingly disparate, highlighting the significant advantage of using a constant stress vault. For instance, at a rise ratio of 2, a catenary has twice the stress of a constant stress vault. This knowledge may also prove useful when working with materials other than snow, such as cast adobe or brick adobe soil materials.

4. Snow volumes based on rise ratio

The volume of material can be calculated using either the length and cross-sectional area or the vertical support reactions at the base. The volume of each type of vault, relative to the rise ratio, can be determined by solving for the parameter *a* for catenary and the parameter *b* for constant stress arch, and selecting the cross-sectional area at the apex. The material volume can be expressed as:

$$V = k_i A_l l, \tag{27}$$

where k_i is the volume coefficient of the corresponding vault type, A_l is the cross-sectional area at the apex and *l* is the span length.

Figure 7 shows the snow volumes in different vault types in relation to the rise ratio. The volume coefficient for the parabolic vault is k_{pvol}, for the catenary it is k_{cvol}, and for the constant stress vault it is k_{csvol}. Figure 7 is divided into two parts due to the volume of the constant stress vault changes significantly as the height of the vault increases. As the height decreases, the volumes of the catenary and constant stress vaults approach the volume of the parabola.

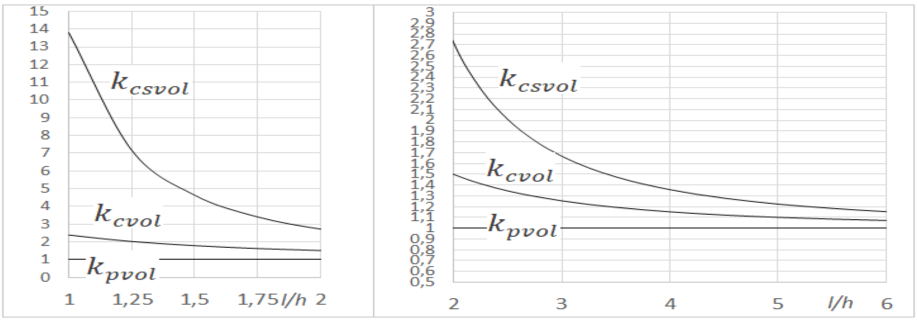

Figure 7: Volume coefficients k_{pvol}, k_{cvol}, and k_{csvo}, according to the rise ratios

5. Vault types compared by span length

5.1. Illustrative comparison of the arch types

In the previous section, we presented the characteristics of stand-alone momentless parabola, catenary, and constant stress vaults. As an example, we carried out comparative calculations and 3D modelling for each type of vault. The snow vaults were designed with equal structural thickness at the crown and experience a maximum compressive stress of -0.15 MPa with a rise ratio of 2.0. Figure 8 shows the shapes and sizes of the calculated vaults in the same scale.

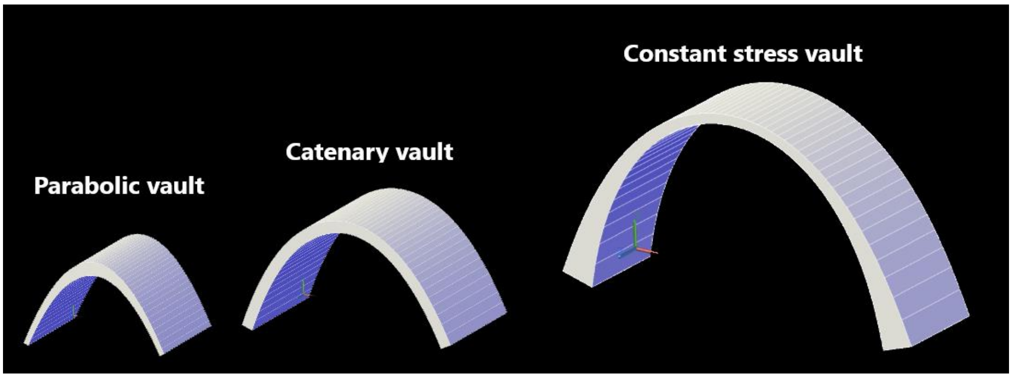

Figure 8: Comparison of the dimensions of snow vaults modelled at the same scale.

With a unit weight of 0.006 MN/m^3 for snow, the span lengths for parabolic, catenary, and constant stress vaults are calculated to be 20.000 m, 30.911 m, and 64.624 m, respectively. This indicates that the constant stress vault is superior to the corresponding parabolic and catenary vaults.

5.2. Rise ratio and the shape of constant stress arch

Figure 9 shows the vaults with rise ratios of 1.5, 2.0, 4.0 and 6.0 when the unit weight of snow is 0.006 MN/m^3. The figure shows also the stresses in the vaults. For a constant stress vault, the span ratio should not be below approximately 0.95 to prevent the soffits from intersecting.

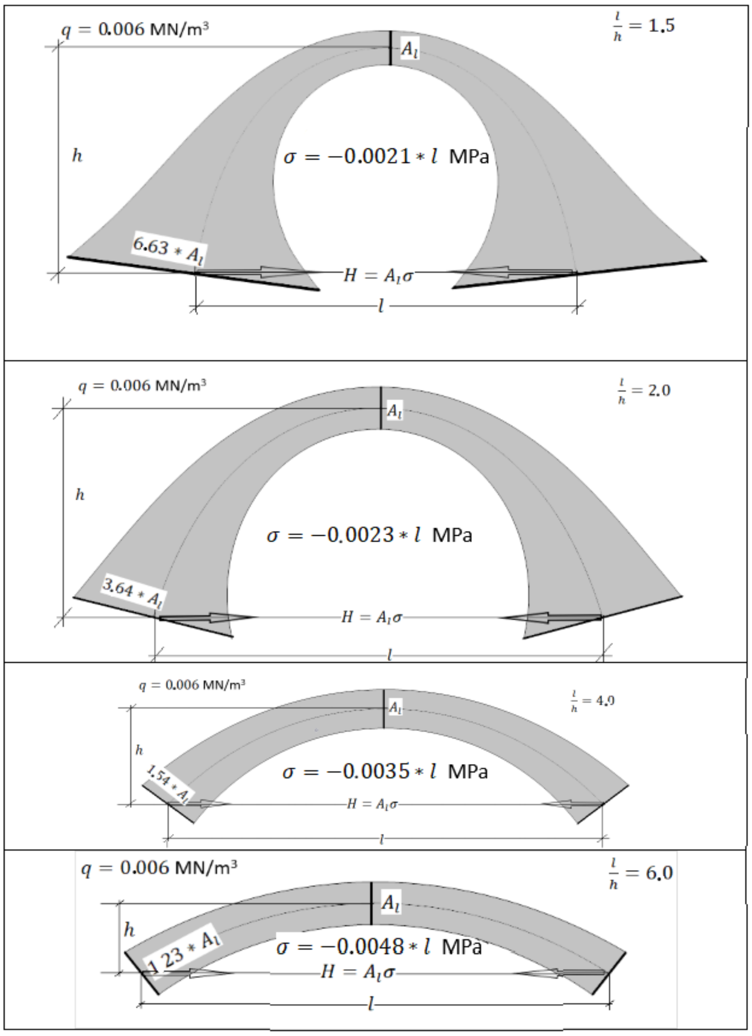

Figure 9: Examples of constant stress snow vaults with rise ratios of 1.5, 2.0, 4.0, and 6.0.

5.3. Comparison of extreme span lengths for parabola, catenary and constant stress vaults

Figure 10 displays the maximum span lengths of the freestanding parabolic, catenary, and constant stress vaults based on the maximum compressive stress. The critical point for the parabolic and catenary vaults is at the base, while for the constant stress vault, the stress remains constant along the entire vault. The assumed compressive stress limit of the snow is -0.2 MPa, and the unit weight of snow used is 0.006 MN/m^3. The stability of the structure is unspecified. The arch crown thickness can be chosen arbitrarily as it does not affect stress magnitude. The calculation results describe the situation at time $t = 0$. In this case, we assume that the snow compression is eliminated by increasing the horizontal force of the arch to match the horizontal force of the uncompressed arch.

The maximum span length of parabolic vault according to the height ratio can be solved from Equation (6). The maximum span is obtained with a rise ratio *l/h* of 4.0. The corresponding maximum span for the catenary is obtained by applying Equation (10) by solving the parameter *a* for corresponding rise relations. The maximum span is obtained with a rise ratio of 2.96, as shown earlier. The corresponding maximum span length of the constant stress arch can be solved by applying Equation (19). The ultimate span is obtained by applying Equation (26).

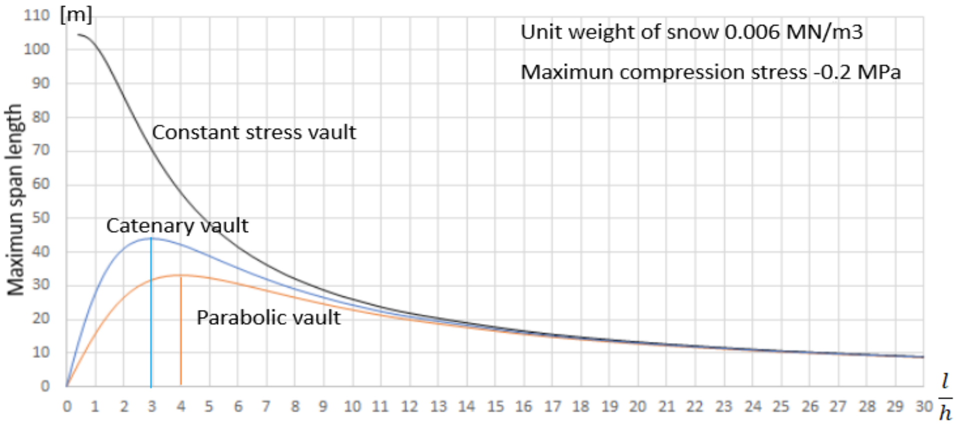

Figure 10. Calculated comparative maximum span lengths of the parabolic, catenary and constant stress snow vaults. Elastic and viscous compression has been eliminated.

Figure 10 shows the differences between the vaults. The difference increases as the height of the arch is increased. As the height is reduced, the catenary and constant stress vault approaches the parabolic vault.

6. Discussions

Snow vault structures require the elimination of elastic and viscous compression of snow to achieve the ideal shape. This can be achieved temporarily by increasing the thrust force of the vault through the installation of hydraulic jacks or reducing the distance between the supports. It may become necessary to increase the thrust again as compression increases over time.

Snow vault is highly sensitive to rapid settlement due to its viscous behaviour, which causes a change in shape and result in bending and shear forces that may make it unsafe to use. Computational methods can be used to calculate viscous deformations (RIL [3]). The viscosity of snow is heavily dependent on its density. When constructing vault structures, it is advisable to use the densest snow available (Ryynänen [4]). The settlements shall be monitored during its life time. Additional research is required.

7. Summary

The constant stress vault is considered the most efficient type of snow vault. However, current instructions for snow structures do not yet recognize its superiority over other vault types, particularly when the height of the vaults is increased.

The calculations show that stress levels in constant stress vaults are minimized, allowing for the construction of reasonable spans using materials with low compression strength. This knowledge could potentially open up the possibility of building temporary, affordable housing using adobe soil material or recycled materials that are locally available. However, additional research is required.

In snow vaults, the heaviest load is typically the dead load. The shape of the vault significantly affects its load-bearing capacity. Settlements should be monitored, and the bearing capacity calculated based on the altered shape.

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