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Lightweight and Floating: Optimizing the Material Usage and Force Flow of Spatial Structures

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Abstract

This work presents a method for designing efficient tensegrity structures. It uses a discrete topology optimization algorithm to analyze candidate structural members within a ground structure. The problem is formulated as a mixed-integer linear program to provide binary choices about which candidate members are selected in the final design and whether each member is undergoing tensile or compressive forces.

Keywords: topology optimization, tensegrity structures, ground structure, mixed-integer optimization.

1. Introduction

Long-span space truss and shell structures present exciting design opportunities. This includes the possibility of creating tensegrity structures. These designs isolate thick compression members by surrounding them with thin tension members on both ends, providing the appearance of floating compression members. The Kurilpa bridge is an example of a tensegrity-inspired design. It is shown in Fig. 1. The horizontal members running perpendicular to the deck are seen to have this floating appearance.

Figure 1: The Kurilpa Bridge in Brisbane, Australia, is a tensegrity-inspired design [1].

To create architecturally distinctive tensegrity designs, it is critical to control the force flow through the structure. Topology optimization lends itself nicely to this problem by providing structurally efficient, lightweight design solutions. Traditional continuum formulations of topology optimization generally seek to maximize the stiffness of a structural design while constraining the material usage [2]. However, modeling long spans as a continuum requires lots of elements. This can make efficient computation challenging and make it difficult to identify constructible structural solutions [3], [4]. Using a discrete approach to topology optimization provides a remedy to this issue. In discrete topology optimization, larger truss and shell elements are used to construct a ground structure of candidate members that fill the design space [5]. Fig. 2 illustrates the difference between these two methods for a simple example from Bendsoe and Sigmund [2].

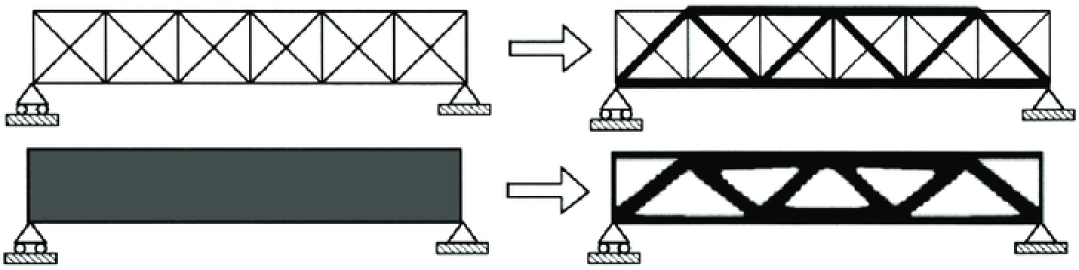


Figure 2: Discrete (top) versus continuum (bottom) topology optimization [2].

This work models ground structure design spaces using a mixed-integer formulation. Previous work has shown that adding mixed-integer variables offers exciting new opportunities for optimization results [6], [7]. An array of materials with different structural properties can be considered by assigning binary on or off values for each material at each candidate member. This can create hybrid designs that provide unique results to a design problem. By leveraging the specific properties of each material, the algorithm can generate designs with intelligent force flows, such as those needed to create a truss connectivity that fulfills the tensegrity definition.

While mixed-integer problems have historically been avoided in topology optimization due to their solving complexity, Kanno [8] presents a method for designing tensegrity structures using a mixed-integer approach. Kanno's formulation solves a minimum compliance problem instead of the minimum material problem addressed by Nanayakkara et al. [9] and this paper. The formulation seen in this work presents an intuitive way to address the multiple element-type problem. Two different materials will be used in the current problem. All tension members will be designed with steel, while all compression members will be designed with timber. Additionally, this work implements Gurobi, an efficient commercial solver for mixed-integer linear programs, to reduce the solving time of the problem.

2. Topology Optimization Formulation

He, Gilbert, and Song [10] propose a formulation for a minimum material discrete topology optimization problem that serves as the basis for this work. Their formulation is a linear program utilizing continuous variables and considering one material. This formulation is expanded upon to generate the mixed-integer linear problem defined in Eq. (1).

There are three sets that index the variables and constants in Eq. (1). The first set, indexed by i, represents each candidate member in the design space. The total number of members in the design space is denoted by N. The second set, indexed by k, represents each structural degree of freedom in the problem. Since this problem implements a truss ground structure, each node at the end of a truss has three degrees of freedom in three-dimensional spaces. Thus, each truss member i has six structural degrees of freedom. The total number of structural degrees of freedom for all members is denoted by

N_F. The third set, indexed by j, represents each material. M denotes the total number of materials considered in the problem. This paper considers two total materials, timber and steel.

This problem includes four design variables. The continuous variable A_{ij} represents the final design cross-sectional area for each candidate truss member i and material j. The continuous variable a_{ij} represents the intermediate optimization cross-sectional area for each candidate truss member i and material j. The continuous variable q_i represents the force going through each candidate truss member i. The binary variable z_{ij} represents if a candidate truss member i and corresponding material j appear in the final design.

$$\underset{A,\,a,\,q,\,z}{\text{minimize}} \quad \sum_{i=1}^{N}\sum_{j=1}^{M} l_i A_{ij} C_j \tag{1a}$$

$$\text{subject to} \quad A_{ij} \leq a_j^{max} z_{ij} \qquad \forall i = 1, \dots, N,\ \forall j = 1, \dots, M \tag{1b}$$

$$A_{ij} \geq a_{ij} - a_j^{max}\,(1 - z_{ij}) \qquad \forall i = 1, \dots, N,\ \forall j = 1, \dots, M \tag{1c}$$

$$\sum_{i=1}^{N} B_{ki} q_i = F_k \qquad \forall k = 1, \dots, N_F \tag{1d}$$

$$q_i \geq -\sum_{j=1}^{M} A_{ij} \sigma_j^c \qquad \forall i = 1, \dots, N \tag{1e}$$

$$q_i \leq \sum_{j=1}^{M} A_{ij} \sigma_j^t \qquad \forall i = 1, \dots, N \tag{1f}$$

$$\sum_{j=1}^{M} z_{ij} \leq 1 \qquad \forall i = 1, \dots, N \tag{1g}$$

$$a_{ij} \geq a_j^{min} \qquad \forall i = 1, \dots, N,\ \forall j = 1, \dots, M \tag{1h}$$

$$a_{ij} \leq a_j^{max} \qquad \forall i = 1, \dots, N,\ \forall j = 1, \dots, M \tag{1i}$$

$$A_{ij} \geq 0 \qquad \forall i = 1, \dots, N,\ \forall j = 1, \dots, M \tag{1j}$$

$$z_{ij} \in \{0, 1\} \qquad \forall i = 1, \dots, N,\ \forall j = 1, \dots, M \tag{1k}$$

The objective function, defined in Eq. (1a), seeks to minimize the material usage of the design. In this equation, the constant l_i represents the length of each candidate truss member i. The constant C_j represents the cost per unit volume of each material j. The linking constraints, defined in Eq. (1b) and Eq. (1c), map the intermediate optimization cross-sectional areas to the final design cross-sectional areas. They also ensure that z_{ij} is only set to one if a candidate truss member i and material j are selected for the final design. The equilibrium constraint, defined in Eq. (1d), ensures that all of the forces in the final design are balanced. The constant B_{ki} represents an entry in the equilibrium matrix for the associated structural degree of freedom k and candidate truss member i. The constant F_k represents the magnitude and direction of an applied force at the structural degree of freedom k. For a structural degree of freedom k where there is no applied force, the corresponding entry F_k is zero. The stress constraints, defined in Eq. (1e) and Eq. (1f), ensure that no candidate truss member is over-stressed. The constants σ_j^c and σ_j^t represent the compressive and tensile stress limits of each material j, respectively. Since this is a linear elastic problem assuming small deformations, the largest value that can be used is the material's yield stress. To ensure the design is conservative, this can be reduced by a factor of safety. The material constraint, defined in Eq. (1g), ensures that a maximum of one material is assigned to a candidate truss member i. This prevents the design from having members with a mixture of properties from multiple materials. The material area constraints, defined in Eq. (1h) and Eq. (1i), ensure that the area selected for a candidate truss member is between the minimum and maximum specified area for each material j. These constants are defined by a_j^{min} and a_j^{max}, respectively. The constraint defined in Eq. (1j) ensures that no final design cross-sectional areas are negative.

To design tensegrity structures, the force flow must be controlled to prevent more than one compression member from connecting to the same node as other compression members entering from different planes and angles. This approach differs from previous formulations because it retains the no-overlap total connectivity ground structure scheme, which reduces the total number of design variables in the problem. This concept is illustrated in Fig. 3, where thick brown lines represent compression members and thin gray lines represent tension members. The far left compression member is composed of two elements that are allowed to share a node because they lie in the same line.

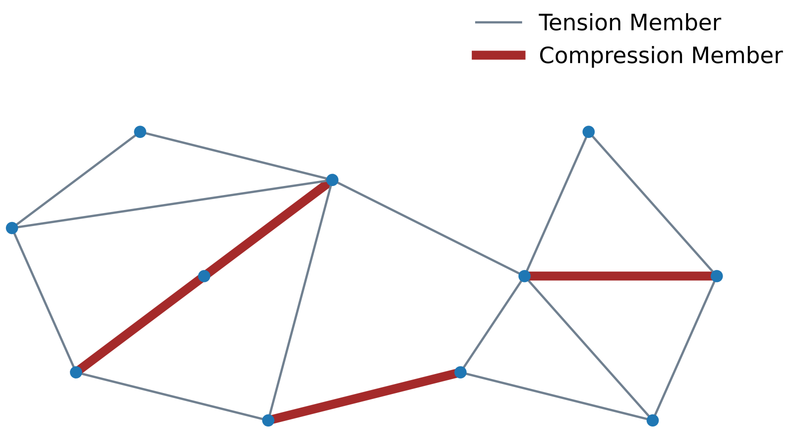


Figure 3: Conceptual design of a tensegrity truss layout.

To achieve tensegrity designs, the problem is expanded to include a fifth design variable, $z_i^{(2)}$. This binary variable is set to one when a candidate truss member selected for the design is in compression and is set to zero otherwise. This variable differs from z_{ij} in that it serves as a force identifier as opposed to a candidate truss member selection identifier. Since the sign function is not linear, this variable is introduced to determine the sign of the force in a member and preserve the mixed-integer linear form of the problem. The constraints in Eq. 2 are added to enforce tensegrity design requirements.

$$q_i \geq -\sigma_{max}^c a_{max}^{max} z_i^{(2)} \qquad \forall i = 1, \dots, N \tag{2a}$$

$$q_i \leq \left(\sigma_{max}^t a_{max}^{max} + \frac{\sigma_{max}^t a_{max}^{max} \varepsilon}{1-\varepsilon}\right)\left(1 - \varepsilon - z_i^{(2)}\right) \qquad \forall i = 1, \dots, N \tag{2b}$$

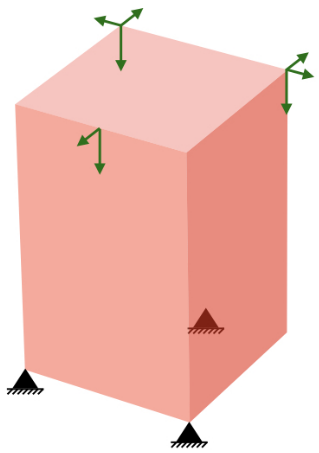
$$\sum_{i=1}^{N} \Gamma_{xi}^{(2)} z_i^{(2)} \leq N \qquad \forall x = 1, \dots, N \tag{2c}$$

In these equations, the set of candidate truss members is indexed by x in addition to i. Eq. 2a and Eq. 2b serve as linking constraints for the binary force variable. They ensure that $z_i^{(2)}$ is only set to one if a candidate truss member is selected for the final design and undergoing compression. The constants σ^c_{max}, σ^t_{max}, and a^{max}_{max} represent the largest allowable compressive stress, largest allowable tensile stress, and largest allowable cross-sectional area across all design materials, respectively. The constant ε is a user-defined small number. It is added to the formulation to ensure that the binary force variable does not take on a value of one when the force in a member is exactly zero, which is the case for candidate truss members not selected for the final design. The tensegrity violation constraint, defined in Eq. 2c, determines which members can exist in the final design to enforce tensegrity principles. The constant Γ_{xi} represents an entry in the relational matrix that verifies if the selected candidate truss members coexist in a tensegrity state.

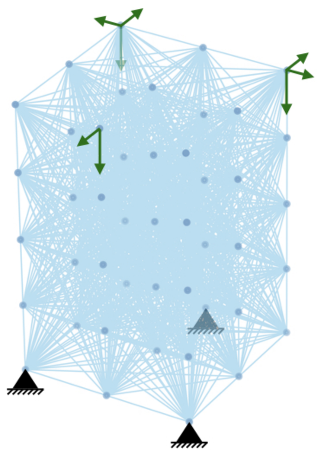
This problem is computationally implemented via a Python script. The code for the equilibrium matrix provided by He, Gilbert, and Song [10] has been expanded to accommodate three-dimensional structures. This equilibrium matrix is denoted as B_{ki} in Eq. (1d). Apart from this, the code written in the Python script is entirely original. It constructs the variables, coefficients, objective function, and constraints defined in Eq. (1) and Eq. (2) to be readily interpreted by Gurobi, the solver used to generate optimized design solutions.

3. Tensegrity Design Example

For simplicity, a relatively small tensegrity design example is examined. The design space, shown in Fig. 4a, occupies a rectangular prism measuring three meters in length, three meters in width, and six meters in height. The design space is supported by pins at two corners of its base, with an additional pin located at the mid-span of the opposite base edge. The design space is loaded at its top in a triangle opposite the supports at the base. The five horizontal loads have a magnitude of 0.45 kN directed outwards and the three vertical loads have a magnitude of 4.5 kN directed downwards. The ground structure, depicted in Fig. 4b, is created by dividing the design space into forty-five nodes. Three nodes are spaced evenly at intervals of one and a half meters along both the length and width dimensions. Five nodes are spaced evenly at intervals of one and a half meters along the height dimension. The ground structure uses a no-overlap total connectivity scheme between the nodes consisting of eight-hundred-thirty-two candidate truss members.



(a) Design space.



(b) Ground structure.

Figure 4: Tensegrity design example boundary and loading conditions.

As mentioned, the algorithm is given a choice between two materials for the design: timber and steel. The compressive strength of the timber is modeled after Douglas fir and set to 8.5 MPa to maintain a factor of safety below the yield strength. The tensile strength of the steel is set to 300 MPa, which also incorporates a factor of safety. The tensile strength of the timber and the compressive strength of the steel are set to zero to ensure that final designs only feature timber in compression and steel in tension. The lower bound area for steel members is set to 19.35 cm^2. The upper bound area for steel members is set to 193.5 cm^2. The lower bound area for timber members is set to 124.2 cm^2. The upper bound area for timber members is set to 322.6 cm^2.

The optimized design result is shown from three different angles in Fig. 5. It is seen to fulfill the tensegrity constraints by preventing multiple compression members that do not lie in the same line from connecting at the same node. The compression members are found to be similar in size, ranging from 124.2 cm^2 to 130.7 cm^2. The steel members take advantage of their higher strength and thus have smaller cross-sectional areas. All steel members take on the minimum cross-sectional area of 19.35 cm^2.

(a) View from above the southeast corner.

(b) View from above the northwest corner.

(c) View from above the southwest corner.

Figure 5: Isometric views of the tensegrity design result.

4. Conclusion

This paper set out to implement a mixed-integer formulation of a discrete topology optimization problem. The algorithm can accommodate the properties of different materials to design lightweight and efficient structures. The formulation also provides the ability to specify desired force flows through a structure. This lends itself nicely to creating unique architectural solutions such as tensegrity designs that give the appearance of floating structural members. By utilizing advanced solvers, formulating highly constrained topology optimization problems as mixed-integer programs is more feasible than ever.

References

[1] J. Smith, *Kurilpa bridge, brisbane*, Last accessed April 13, 2024, 2009. [Online]. Available: https://www.flickr.com/photos/26085795@N02/4082193852.

[2] M. P. Bendsoe and O. Sigmund, *Topology Optimization: Theory, Methods, and Applications*. Springer, 2004, ISBN: 978-3-662-05086-6. DOI: https://doi.org/10.1007/978-3-662-05086-6.

[3] J. V. Carstensen, H. Kim-Tackowiak, and M. Y. Liang, "Improving the manufacturability of highly materially restricted topology-optimized designs with mixed integer linear programming," *Engineering Structures*, vol. 284, p. 115 955, 2023. DOI: https://doi.org/10.1016/j.engstruct.2023.115955.

[4] M. Stolpe and K. Svanberg, "On the trajectories of penalization methods for topology optimization," *Structural and Multidisciplinary Optimization*, vol. 21, pp. 128–139, 2001. DOI: https://doi.org/10.1007/s001580050177.

[5] W. Dorn, R. E. Gomory, and H. Grenberg, "Automatic design of optimal structures," 1964.

[6] H. Fairclough and M. Gilbert, "Layout optimization of simplified trusses using mixed integer linear programming with runtime generation of constraints," *Structural and Multidisciplinary Optimization*, vol. 61, pp. 1977–1999, 2020. DOI: https://doi.org/10.1007/s00158-019-02449-7.

[7] A. Asadpoure, M. Harati, and M. Tootkaboni, "Discrete topology optimization in augmented space: Integrated element removal for minimum size and mesh sensitivity control," *Structural and Multidisciplinary Optimization*, vol. 62, pp. 2615–2627, 2020. DOI: https://doi.org/10.1007/s00158-020-02630-3.

[8] Y. Kanno, "Topology optimization of tensegrity structures under compliance constraint: A mixed integer linear programming approach," *Optimization and Engineering*, vol. 14, pp. 61–96, 2013. DOI: https://doi.org/10.1007/s11081-011-9172-0.

[9] K. I. U. Nanayakkara, L. He, H. E. Fairclough, and M. Gilbert, "A simple layout optimization formulation for load-carrying tensegrity structures," *Structural and Multidisciplinary Optimization*, vol. 62, pp. 2935–2949, 2020. DOI: https://doi.org/10.1007/s00158-020-02653-w.

[10] L. He, M. Gilbert, and X. Song, "A python script for adaptive layout optimization of trusses," *Structural and Multidisciplinary Optimization*, vol. 60, pp. 835–847, 2019. DOI: https://doi.org/10.1007/s00158-019-02226-6.