Proceedings of the IASS 2024 Symposium
Redefining the Art of Structural Design
August 26-30, 2024, Zurich, Switzerland
Philippe Block, Giulia Boller, Catherine DeWolf,
Jacqueline Pauli, Walter Kaufmann (eds.)

Structural Analysis Using the Redundancy Matrix and Graph Theory

David FORSTER[a,*], William F. BAKER[b], Manfred BISCHOFF[a]

[a] University of Stuttgart, Institute for Structural Mechanics
Pfaffenwaldring 7, 70569 Stuttgart
*forster@ibb.uni-stuttgart.de

[b] Skidmore, Owings & Merrill, LLP
224 S. Michigan Avenue, Chicago, IL 60604, USA

Abstract

Structural engineers often want to have a redundant structure where the loss of a member would not lead to structural collapse. For a truss, adding a bar beyond that required for static determinacy renders the structure redundant, but what is the spatial distribution of the static indeterminacy within the individual elements of a framework? Can an additional bar be redundant with several existing bars? Are there truss topologies and geometries that enhance redundancy? The assessment of structures based on such load-independent quantitative measures can be useful in early design stages to achieve an integrative planning process for designers and engineers. The degree of static indeterminacy and in particular its spatial distribution, quantified with the redundancy matrix can be used for assessing structural integrity of a framework. Focusing on structural properties independent of the individual member stiffness, such as geometry and topology, graph theory offers yet another tool to assess structural performance. This paper explores the integration of the Maxwell-Calladine count with the redundancy matrix from theoretical structural mechanics and with contributions of graph theory to explore a deeper understanding of structural redundancy.

Keywords: structural redundancy, static indeterminacy, redundancy matrix, graph theory, structural analysis

1. Introduction

A fundamental question for structural engineers, especially in early design stages, is whether a structural assembly is rigid or not. Here, the term "rigid" refers to being free from kinematic modes with zero stiffness, rather than "infinitely stiff". One answer lies in the degree of static indeterminacy, which quantifies the amount of redundant load-transfer mechanisms as an integer number, introduced by Maxwell in 1864 [1]. This counting rule, however, is not sufficient to assess whether a structure has states of self-stress and kinematic mechanisms at the same time. Therefore, Calladine completed the counting theory with recognition that the Maxwell count equals the number of mechanisms less the number of states of self-stress [2].

One drawback of the counting rules is that they do not provide any information about the spatial distribution of static indeterminacy over the individual elements of a framework. In the group of Klaus Linkwitz at the University of Stuttgart, the so-called redundancy matrix was derived, which quantifies this distribution of the degree of static indeterminacy in a structure [3, 4, 5]. A detailed derivation for truss and beam structures can be reviewed in von Scheven et al. [6]. The redundancy matrix proves feasible as a performance indicator for the design and optimization of structures. Examples with regards to robustness and structural on-site assembly are shown in Forster et al. [7]. In the context of integrative computational design and construction processes, as described in Knippers et al. [8], the redundancy matrix can be used to quantify the individual element's importance in coreless filament-wound structures [9].

Besides these well-established concepts for structural engineers, the rigidity and redundancy of frameworks can also be analyzed with the help of graph theory, a branch of mathematics, elaborating on the connectivity of frameworks [10]. Bolker and Crapo present a special way to solve the problem of placing braces in a rectangular framework by encoding the diagonal truss elements with a bipartite graph [11, 12]. However, the resulting graph only includes information about the spatial distribution of the braces in the framework and lacks information about the redundancy of the individual elements. The contribution of Achi and Tibert [13] proposes to use graph theory and the redundancy distribution for interdisciplinary exchange to assess structural behavior by providing an overview of existing literature in the two fields.

This contribution presents the analysis of frameworks combining the Maxwell-Calladine count, the redundancy matrix and graph theory. The representation of frameworks using bipartite graphs is enriched with information about the redundancy distribution. In Section 2, we introduce the fundamentals of matrix structural analysis necessary to calculate the redundancy matrix and to understand the counting rules as well as the basics of graph theory. In Section 3, the interaction of the above-mentioned structural assessment methods is showcased with truss examples. Section 4 summarizes the work and points out open research questions.

2. Matrix structural analysis and graph theory

This section describes the fundamentals of matrix structural analysis, the redundancy matrix, the counting rules for the determination of static and kinematic indeterminacy, and graph theory. We follow the notation in von Scheven et al. [6]. For a more in-depth study of matrix structural analysis and details on the fundamental subspaces of the equilibrium matrix, the reader is referred to Livesley [14] as well as Pellegrino and Calladine [15]. We consider a discrete truss structure with n_e elements, n_n nodes, n_c kinematic constraints and n_d unconstrained nodal displacements. The kinematic equations, describing the relation between nodal displacements $\mathbf{d} \in \mathbb{R}^{n_\mathrm{d}}$ and element elongations $\Delta\mathbf{l} \in \mathbb{R}^{n_\mathrm{e}}$ via the compatibility matrix $\mathbf{A} \in \mathbb{R}^{n_\mathrm{e} \times n_\mathrm{d}}$ can be written as

$$\mathbf{A}\mathbf{d} = \Delta\mathbf{l}. \tag{1}$$

The equations of equilibrium, relating the external loads $\mathbf{f} \in \mathbb{R}^{n_\mathrm{d}}$ via the equilibrium matrix $\mathbf{A}^\mathrm{T} \in \mathbb{R}^{n_\mathrm{d} \times n_\mathrm{e}}$ to the internal forces $\mathbf{N} \in \mathbb{R}^{n_\mathrm{e}}$ read

$$\mathbf{A}^\mathrm{T}\mathbf{N} = \mathbf{f}. \tag{2}$$

To complete the governing equations of linear elasto-statics, the material matrix $\mathbf{C} \in \mathbb{R}^{n_\mathrm{e} \times n_\mathrm{e}}$, containing the individual element's member stiffness EA/l on its main diagonal, is introduced. Thus, the elastic material law can be written as

$$\mathbf{N} = \mathbf{C}\Delta\mathbf{l}_\mathrm{el}. \tag{3}$$

If a structural system is statically determinate, the rank of the equilibrium matrix equals the number of elements. In this case, the internal forces can be calculated by solely using the equilibrium equation (2). The definition of the degree of static indeterminacy n_s dates back to Maxwell [1] and the counting rule is therein defined as

$$n_\mathrm{s} = n_\mathrm{e} + n_\mathrm{c} - 2n_\mathrm{n} \tag{4}$$

for plane trusses. If a bar is added to an initially statically determinate system, equation (4) gives $n_\mathrm{s} = 1$. The system becomes statically indeterminate and contains a possible state of self-stress due to the additional bar, meaning that in this potential stress state the internal forces are in equilibrium without any external load. All states of self-stress in a system can be found in the nullspace of $\mathbf{A}^\mathrm{T}$ [15]. On the other hand, if one bar is removed from an initially statically determinate structure, a kinematic mechanism will be present and the Maxwell count results in $n_\mathrm{s} = -1$. The nullspace of $\mathbf{A}$ spans the subspace of kinematic mechanisms [15]. Thus, the potential states of self-stress and the kinematic mechanisms are properties of the structure only using information about topology and geometry, independent of cross sectional stiffness.

Due to the interplay of self-stress states and kinematic mechanisms, which can both be present in a structure at the same time, the Maxwell count is extended by Calladine [2] to

$$n_\mathrm{s} - n_\mathrm{m} = n_\mathrm{e} + n_\mathrm{c} - 2n_\mathrm{n} \tag{5}$$

for plane trusses, n_m being the degree of kinematic indeterminacy. This extension completes the assessment of structures in terms of the degree of static and kinematic indeterminacy. Figure 1 shows two different truss examples and the respective results for the analysis with the counting rules. The lower truss is statically indeterminate by degree two and contains two states of self-stress, which are both illustrated in the second column using red and blue colors for tension and compression, respectively. The thickness of the lines represents the magnitude of the member force. In contrast to this, the upper truss has a kinematic mechanism, shown in grey in the first column, and a degree of static indeterminacy of one. In this case, equation (4) would fail to recognize the kinematic mechanism of this special geometry. The self-stress state for the upper truss, which forms only in the two vertical members on the left, is also shown in the second column.

Equations (4) and (5) both result in a single integer number, which provides little insight into the load-bearing behavior of the structure, especially in structures with a large number of elements. Therefore, an open question remains when applying the above-mentioned counting rules: How is the static indeterminacy distributed within the structure? This question can be answered using the redundancy matrix, derived in the group of Klaus Linkwitz [3, 4, 5]. The redundancy matrix $\mathbf{R} \in \mathbb{R}^{n_\mathrm{e} \times n_\mathrm{e}}$ is described as a mapping of initial, prescribed elongations $\Delta\mathbf{l}_0$ to negative elastic elongations $\Delta\mathbf{l}_\mathrm{el}$ as

$$\Delta\mathbf{l}_\mathrm{el} = -(\mathbf{1} - \mathbf{A}\mathbf{K}^{-1}\mathbf{A}^\mathrm{T}\mathbf{C})\Delta\mathbf{l}_0 = -\mathbf{R}\Delta\mathbf{l}_0, \tag{6}$$

$\mathbf{K} = \mathbf{A}^\mathrm{T}\mathbf{C}\mathbf{A}$ being the elastic stiffness matrix of the system. As opposed to the self-stress states and kinematic mechanisms, the redundancy distribution is calculated using information about the topology and geometry as well as the elastic properties of the structure, namely the cross-sectional stiffness in the case of trusses, as it quantifies the constraint of the structure on the individual elements. If the structure has a kinematic mechanism ($n_\mathrm{m} > 0$), the stiffness matrix is singular and thus not invertible. Therefore, the generalized inverse must be used in equation (6), as presented by Chen et al. [17]. The main diagonal entries of the redundancy matrix quantify the spatial distribution of the static indeterminacy. Thus, the trace sums up to the degree of static indeterminacy [6]:

$$\mathrm{tr}(\mathbf{R}) = n_\mathrm{s}. \tag{7}$$

The maximum redundancy for a single truss element is one, meaning that this element can be removed without altering the load-bearing-behavior of the structure. The minimum redundancy value of zero means that the non-redundant truss element is indispensable for structural integrity and a removal of such an element would result in a (partial) collapse of the structure. The image of the transpose of the

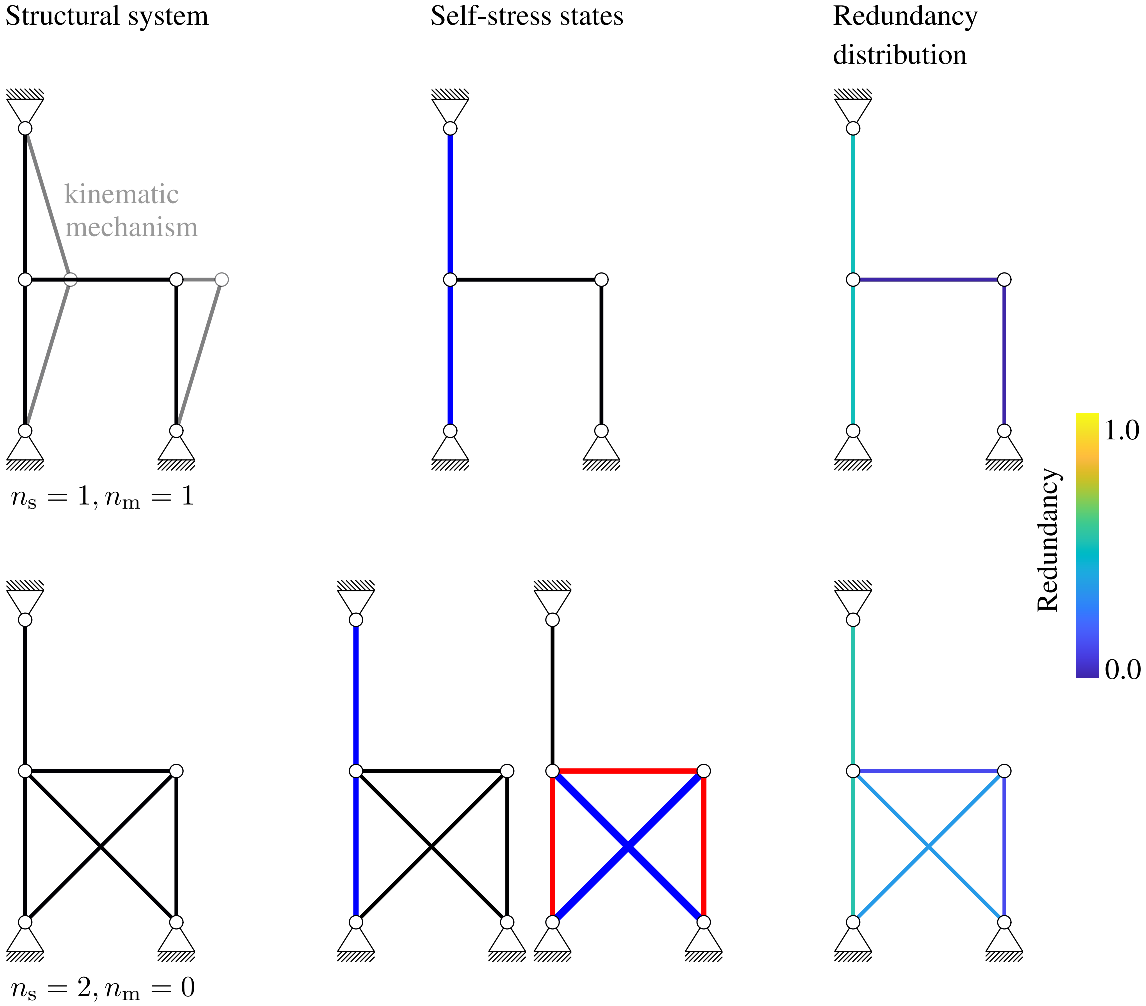

Figure 1: Two different structural systems and the degree of static and kinematic indeterminacy shown in column one. The kinematic mechanism of the first structure drawn in grey. The respective self-stress states shown in the second column (red and blue indicating compression and tension; magnitude of force scaled with line thickness) and the redundancy distribution shown in colorscheme in the third column. The example is adapted from Pellegrino [16].

redundancy matrix, $\mathrm{im}(\mathbf{R}^{\mathrm{T}})$, is equal to the nullspace of the equilibrium matrix [18]. This means, that the states of self-stress can also be identified with the redundancy matrix. The redundancy distribution of the two truss examples in Figure 1 are shown in colorscheme on the right. One can see in dark blue color in the upper truss, that the elements that are not stressed in the self-stress state have zero redundancy. In other words, they are statically determinate. In the lower example, each element has a certain redundancy, meaning that failure of any one element would not lead to structural collapse in this case. The redundancy of an individual element increases with the number of available alternative load paths and the stiffness associated with these.

Another way to assess structures comes with graph theory, a branch of mathematics that can in general be used to model many kinds of relationships. In the context of structural mechanics, the so-called structure graph describes how a framework is connected and thus the relationship between the nodes. For a detailed and extensive introduction to this field, we refer to the textbooks of Graver [10] and Wilson [19]. In order to apply graph theory in the context of structural assessment, some necessary

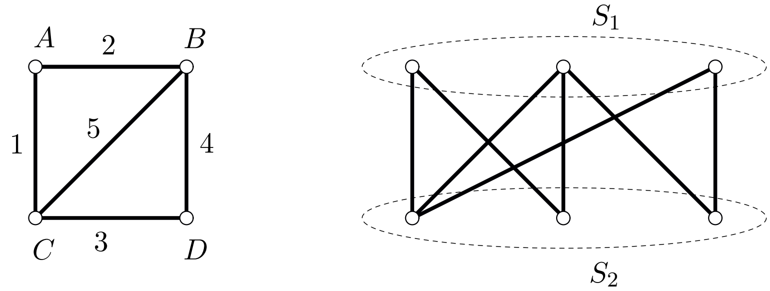

Figure 2: A connected graph with vertices A to D and edges 1 to 5 is shown on the left. A bipartite graph with the two vertex sets S_1 and S_2 is shown on the right, indicating the partitioning of the vertices.

definitions will be recapitulated. A graph (V, E) is defined as a pair of vertices V and edges E. Therein, the edges describe the collection of pairs of vertices. A graph (V, E) is defined as connected, if no partition into two nonempty sets A and B exists ($V = A \cup B, A \cap B = \varnothing, A \neq \varnothing, B \neq \varnothing$). Figure 2 shows on the left a graph with four vertices (A to D) and five edges (1 to 5). It is a connected graph, since no partition of the vertices into two nonempty sets exists. In other words, starting at a random vertex, one can reach every other vertex by using the available edges. Furthermore, the notion of a bipartite graph is introduced, which can be used for encoding bracing elements in a truss grid, as shown in Bolker and Crapo [12]. A graph (V, E) is called bipartite, if its vertex set can be partitioned into two sets S_1 and S_2 of disjoint vertices. This means that each edge E has an endpoint in S_1 and an endpoint in S_2. Such an exemplary bipartite graph is shown in Figure 2 on the right.

3. Structural assessment beyond stress and strain

In this section, the structural assessment of trusses going beyond the typical performance indicators, like stresses, strains or displacements, is presented by using an interplay of the Maxwell-Calladine count, the redundancy distribution and graph theory. For this purpose, a special grid bracing problem is shown, in which the bracing of the structure can be abstracted with the help of graph theory, as first presented by Bolker and Crapo [11, 12] and described in detail in Graver [10]. We consider a $r \times c$ rectangular grid of truss members, where r and c refer to the number of rows and columns, respectively. The structure is constrained by a fixed support and a roller support in a statically determinate manner.

An eigenvalue decomposition of the stiffness matrix of the grid leads to $(r + c - 1)$ zero eigenvalues, meaning that at least $(r+c-1)$ diagonal bracings are necessary to eliminate the kinematic mechanisms. Bolker and Crapo describe, how the bracing elements can be encoded with a bipartite graph by numbering the rows and columns of the grid. We will refer to the numbering as rows r_i and columns c_k. With the help of the associated bipartite graph, the bracing can be identified to be rigid if and only if this graph is connected [10]. This means, that a path connecting all vertices must exist as described in Section 2. With this, one can assess the structure in such a way that the existence of kinematic mechanisms and redundant bracing elements are easy to locate. Nevertheless, little insight into the load-bearing behavior of the structure and the quantitative impact of different bracing options are given. Therefore, the information about the redundancy of the bracings, as described in Section 2, is added to the bipartite graph in colorscheme. This enriches the visual encoding of the bracings via a bipartite graph and quantifies the structural importance of the individual bracings.

Figure 3 shows three different solutions of the bracing problem. On the left, the minimal number of bracings is used to render the structure rigid, as the associated bipartite graph is connected. As it can be seen in colorscheme of the graph below the structure, the bracings are all non-redundant. Removing a bracing would lead to a kinematic mechanism. In the center of Figure 3, an additional brace is added,

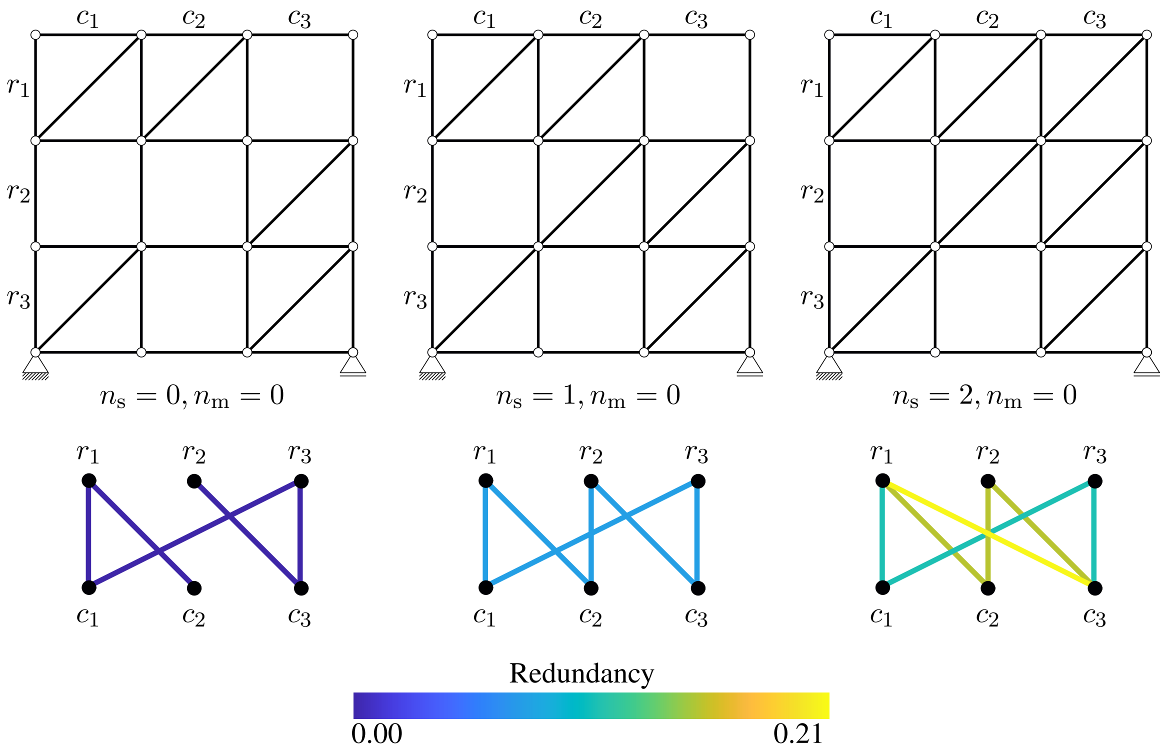

Figure 3: Three different options of bracing the 3×3 grid with the associated bipartite graph below. The edges represent the bracing elements of the associated column and row of the grid. The colorscheme indicates the redundancies of the diagonal elements only.

represented in the bipartite graph by connecting r_2 and c_2. This additional bracing is redundant with every other member, meaning that any one of the diagonals can be removed without the structure collapsing subsequently. The redundancies are distributed homogeneously throughout the diagonals, which indicates a robust design, as described by Forster et al. [20]. On the right of Figure 3, another bracing is added, which increases the degree of static indeterminacy by one and thus, the redundancies of the diagonals are distributed from a minimum of 0.11 to a maximum of 0.21. The lastly added diagonal on the top right (r_1, c_3) is the most redundant one, meaning that its removal would have the least impact regarding structural performance.

Figure 4 shows a bracing that also uses five diagonals, like the statically determinate version above. However, their specific placement results in a structure that is kinematic and has one state of self-stress. According to equation (5) it is $n_s = n_m = 1$. On the left, the bracing and the associated bipartite graph are shown, indicating that the four outer diagonals are equally redundant. Since the bipartite graph is not connected, the structure has a kinematic mechanism, which can be seen on the top right. On the bottom right, the state of self-stress is shown. The diagonal bracing in the center of the grid is in an unstressed state, which relates to the fact that this member is statically determinate, as can be seen in dark blue color on the bipartite graph, and thus it cannot be prestressed.

The investigation of the trusses has shown that by integrating redundancy analysis and graph theory, information about the structural importance of individual elements can be seen right away in the associated graph. Information about the distribution of constraint within a structure is added to the purely topological representation, adding insight into the load-bearing behavior of the structure to the graphical encoding. Moreover, the involvement of the individual elements in states of self-stress can be evaluated through the graphical representation.

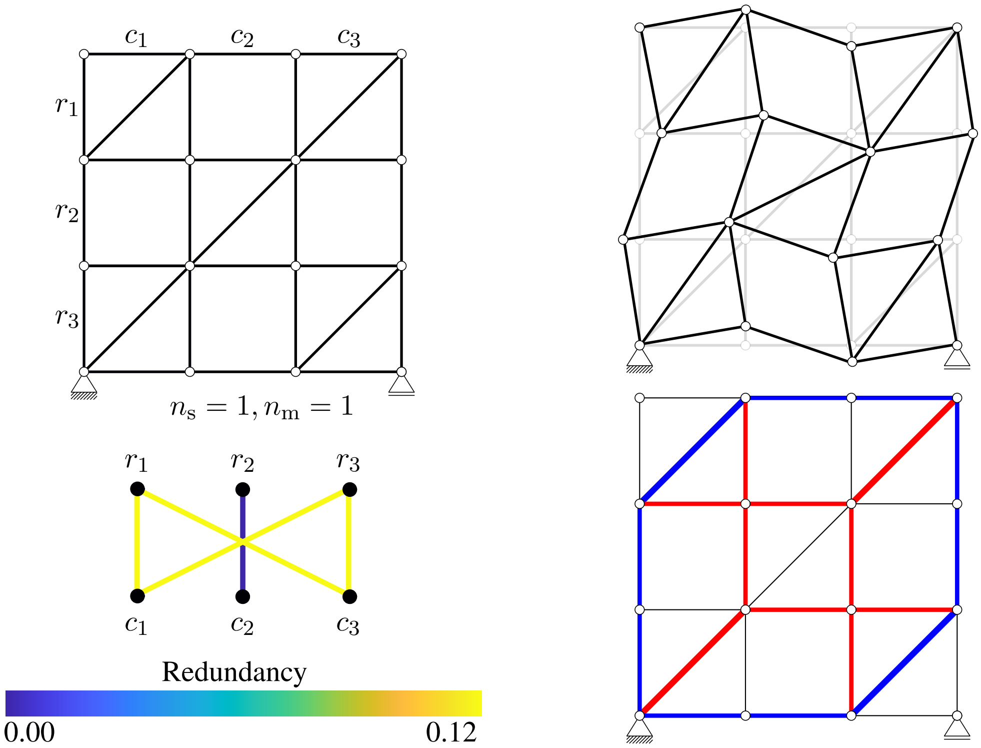

Figure 4: A special option of bracing the 3×3 grid with the associated bipartite graph below. The edges represent the bracing elements of the associated column and row of the grid. The colorscheme indicates the redundancies of the diagonal elements only. Kinematic mechanism and state of self-stress shown on the right.

4. Conclusion and outlook

The Maxwell counting rule lacks information about the distribution of redundancy in statically indeterminate structures. By adding this information with the redundancy matrix, insight into the load-bearing behavior of structures can be improved. The redundancy matrix for load bearing structures, quantifying the distributed static indeterminacy, was first derived by the group of the Geodesist Klaus Linkwitz in Stuttgart. The present contribution shows an integration of the Calladine-Maxwell counting rule, the redundancy distribution and graph theory to assess structural performance. By enriching the graphical encoding of a grid bracing in a bipartite graph with the redundancies of the individual elements, the structural importance of the bracing elements can now be evaluated on a quantitative basis. This leads to a representation of the structural configuration beyond only the topology by means of the connectivity of the graph. As an extension of the present work, the application of a graphical representation of beam elements and the relation between the Airy stress function and the redundancy distribution is current work of the authors.

Acknowledgments

This research was supported by the Deutsche Forschungsgemeinschaft (DFG; German Research Foundation) under Germany's Excellence Strategy – EXC 2120/1 – 390831618.

References

[1] J. C. Maxwell, "On the calculation of the equilibrium and stiffness of frames," *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, vol. 27, no. 182, pp. 294–299, 1864, ISSN: 1941-5982. DOI: 10.1080/14786446408643668.

[2] C. Calladine, "Buckminster Fuller's "Tensegrity" structures and Clerk Maxwell's rules for the construction of stiff frames," en, *International Journal of Solids and Structures*, vol. 14, no. 2, pp. 161–172, 1978. DOI: 10.1016/0020-7683(78)90052-5.

[3] K. Linkwitz, *Fehlertheorie und Ausgleichung von Streckennetzen nach der Theorie elastischer Systeme* (Dissertation, Universität Stuttgart, Deutsche Geodätische Kommission bei der Bayerischen Akademie der Wissenschaften, Reihe C: Dissertationen Heft Nr. 46). München: Beck Verlag, 1961.

[4] J. Bahndorf, *Zur Systematisierung der Seilnetzberechnung und zur Optimierung von Seilnetzen* (Dissertation, Universität Stuttgart, Deutsche Geodätische Kommission bei der Bayerischen Akademie der Wissenschaften, Reihe C: Dissertationen Heft Nr. 373). München: Beck Verlag, 1991, ISBN: 3 7696 9420 1.

[5] D. Ströbel, *Die Anwendung der Ausgleichungsrechnung auf elastomechanische Systeme* (Dissertation, Universität Stuttgart, Deutsche Geodätische Kommission bei der Bayerischen Akademie der Wissenschaften, Reihe C: Dissertationen Heft Nr. 478). München: Beck Verlag, 1997, ISBN: 3 7696 95186.

[6] M. von Scheven, E. Ramm, and M. Bischoff, "Quantification of the redundancy distribution in truss and beam structures," *International Journal of Solids and Structures*, vol. 213, pp. 41–49, 2021, ISSN: 0020-7683. DOI: https://doi.org/10.1016/j.ijsolstr.2020.11.002.

[7] D. Forster, F. Kannenberg, M. von Scheven, A. Menges, and M. Bischoff, "Design and optimization of beam and truss structures using alternative performance indicators based on the redundancy matrix," in *Proceedings of Advances in Architectural Geometry 2023, October 6-7, 2023, Stuttgart*, 2023, pp. 455–466. DOI: 10.1515/9783111162683-034.

[8] J. Knippers, C. Kropp, A. Menges, O. Sawodny, and D. Weiskopf, "Integrative computational design and construction: Rethinking architecture digitally," en, *Civil Engineering Design*, vol. 3, no. 4, pp. 123–135, Sep. 2021. DOI: 10.1002/cend.202100027.

[9] M. Gil Pérez *et al.*, "Data processing, analysis, and evaluation methods for co-design of coreless filament-wound building systems," en, *Journal of Computational Design and Engineering*, vol. 10, no. 4, pp. 1460–1478, Jul. 2023, ISSN: 2288-5048. DOI: 10.1093/jcde/qwad064.

[10] J. E. Graver, *Counting on frameworks: mathematics to aid the design of rigid structures* (Dolciani mathematical expositions no. 25). Washington, DC: Mathematical Association of America, 2001, ISBN: 978-0-88385-331-3.

[11] E. D. Bolker and H. Crapo, "How to brace a one-story building," en, *Environment and Planning B: Planning and Design*, vol. 4, no. 2, pp. 125–152, 1977, ISSN: 0265-8135, 1472-3417. DOI: 10.1068/b040125.

[12] E. D. Bolker and H. Crapo, "Bracing Rectangular Frameworks. I," *SIAM Journal on Applied Mathematics*, vol. 36, no. 3, pp. 473–490, 1979. DOI: 10.1137/0136036.

[13] L. Martinsson Achi and G. Tibert, "A graph theoretical methodology for conceptual design," in *IASS-APCS 2012 : Proceedings of the International Symposium on Shell and Spatial Structures*, 2012. [Online]. Available: https://iass-structures.org/Annual-Symposia.

[14] R. Livesley, *Matrix Methods of Structural Analysis*, en-gb, 2nd ed. Pergamon Press, 1975, ISBN: 978-0-08-018888-1.

[15] S. Pellegrino and C. Calladine, “Matrix analysis of statically and kinematically indeterminate frameworks,” *International Journal of Solids and Structures*, vol. 22, no. 4, pp. 409–428, 1986, ISSN: 00207683. DOI: 10.1016/0020-7683(86)90014-4.

[16] S. Pellegrino, “Structural computations with the singular value decomposition of the equilibrium matrix,” *International Journal of Solids and Structures*, vol. 30, no. 21, pp. 3025–3035, 1993, ISSN: 0020-7683. DOI: https://doi.org/10.1016/0020-7683(93)90210-X.

[17] Y. Chen, J. Feng, H. Lv, and Q. Sun, “Symmetry representations and elastic redundancy for members of tensegrity structures,” *Composite Structures*, vol. 203, pp. 672–680, Nov. 2018, ISSN: 0263-8223. DOI: 10.1016/j.compstruct.2018.07.044.

[18] F. Geiger, “Strukturmechanische Charakterisierung von Stabtragwerken für den Entwurf adaptiver Tragwerke,” de, Dissertation, Bericht 74 / Institut für Baustatik und Baudynamik der Universität Stuttgart, Stuttgart, 2022. DOI: 10.18419/OPUS-12299.

[19] R. J. Wilson, *Introduction to graph theory*, en, 4th edition. Harlow Munich: Prentice Hall, 1996, ISBN: 978-0-582-24993-6.

[20] D. Forster, M. von Scheven, and M. Bischoff, “Alternative Beurteilung von Tragwerken mit Hilfe der Redundanzmatrix,” in *Berichte der Fachtagung Baustatik - Baupraxis 15*, B. Oesterle, A. Bögle, W. Weber, and L. Striefler, Eds., 2024, pp. 67–74, ISBN: 978-3-00-077808-7.