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Investigation on an Analytical Approach for Tendon Layout Optimization Using Strain Energy Minimization

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Abstract

Prestressing concrete elements significantly enhances their structural performance by reducing deformations and improving crack behavior. The tendon layout is typically designed to partially or fully compensate for permanent loads, which, in standard geometries, lead to relatively simple polygonal or parabolic shapes. However, for more complex geometries, such as spatial structures, where greater precision is required, the current approaches may be insufficient.

This paper investigates tendon layout optimization in concrete structures by minimizing strain energy. Starting from the general formulation for strain energy-based layout optimization, an analytical solution for 2-dimensional prestressed concrete beams is sought using methods from the calculus of variations. Given the high complexity of the resulting differential equation, the necessity for a numerical approach becomes evident. When assuming a geometrical simplification, a formula commonly used in engineering practice for defining tendon layouts emerges. This demonstrates an accurate formulation of the problem while simultaneously highlighting the indispensable need for a numerical method.

Keywords: prestressed concrete, prestress force, strain energy, structural optimization, tendon layout.

1. Introduction

In prior research [1, 2], the optimization of tendon layout for beams and spatial structures has been investigated using the minimization of strain energy. These investigations have predominantly adopted a numerical approach, briefly noting the impracticality of an analytical solution. In this paper, the reasons why an analytical solution is not feasible will be detailed.

For the following analysis, it will be assumed that the prestressed concrete beams have a constant rectangular cross-section. This initial focus on a basic scenario is crucial, as it establishes whether an analytical solution is feasible before addressing more complex geometries. The model used in order to optimize the tendon layout is shown in Figure 1. Here, the tendon is located in the XZ-plane of the beam. In the two-dimensional analysis, a parametrization with $t = x$ is assumed, thus the position vector for the tendon is defined as $\boldsymbol{r} = (x, r(x))$, representing the function to be determined. The prestressing force is represented by the vector $\boldsymbol{P}(x)$. A beam fixed at one end and articulated at the other is exemplary shown in Figure 1. However, the analysis that follows is independent of the support conditions, based on the assumption that statically determinate systems will be considered at the time of prestressing. This implies that statically indeterminate systems may also be considered, provided that the prestressing force is induced in a previously statically determinate state (for more details, see [1, 2]).

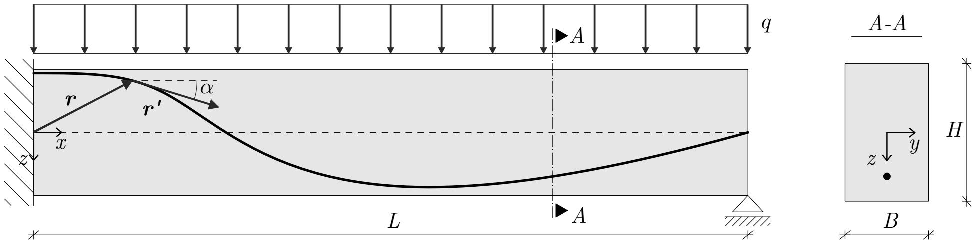

Figure 1: Tendon layout for beam fixed at one end.

1.1. Functional definition

Building on previous investigations [1, 2], the strain energy will be assumed as the functional. Minimization of this functional correlates with enhanced stiffness and a consequent reduction in deformations, aligning with the primary objectives of prestressing. The strain energy can be generally formulated as follows [3]:

$$\mathit{\Pi}_i = \frac{1}{2}\int_V \boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}\, dV = \frac{1}{2}\int_V \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix} \begin{pmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z \end{pmatrix} dV \tag{1}$$

Considering that the focus will be on beams primarily subject to bending, shear stresses and vertical axial stresses will be disregarded in the subsequent analyses. Mathematically, this is expressed by assuming $\sigma_y = \sigma_z = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0$. Consequently, the strain energy can be simplified as follows:

$$\mathit{\Pi}_i = \frac{1}{2E}\int_V \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix} \begin{pmatrix} \sigma_x - \nu(\sigma_y+\sigma_z) & (1+\nu)\tau_{xy} & (1+\nu)\tau_{xz} \\ (1+\nu)\tau_{yx} & \sigma_y - \nu(\sigma_x+\sigma_z) & (1+\nu)\tau_{yz} \\ (1+\nu)\tau_{zx} & (1+\nu)\tau_{zy} & \sigma_z - \nu(\sigma_x+\sigma_y) \end{pmatrix} dV \tag{2}$$

$$= \frac{1}{2E}\int_V \begin{pmatrix} \sigma_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sigma_x & 0 & 0 \\ 0 & -\nu\sigma_x & 0 \\ 0 & 0 & -\nu\sigma_x \end{pmatrix} dV = \frac{1}{2E}\int_V \sigma_x^2\, dV \tag{3}$$

In the context of prestressed concrete beams, the stresses σ_x can be partitioned into a component due to external loading, denoted σ_E, and a component arising from prestressing, represented by σ_P:

$$\Rightarrow \mathit{\Pi}_i = \frac{1}{2E}\int_V (\sigma_E + \sigma_P)^2\, dV \tag{4}$$

The stress field attributable to external loads is governed by the following equation:

$$\sigma_E(x,z) = \frac{M_E(x)}{I_y} z \tag{5}$$

Conversely, the stresses resulting from the prestressing force comprise an axial force component and a moment component. In statically determinate systems, the moment induced by prestressing arises from the horizontal component of the prestressing force, denoted as P_H, and its lever arm relative to the beam's centroidal axis, denoted as $r(x)$. Consequently, the moment due to prestressing is determined as follows:

$$M_P(x) = -P_H(x)\, r(x) \tag{6}$$

Taking into account both the normal force and the moment component from the prestressing force, σ_P can be calculated as follows:

$$\sigma_P(x,z) = \frac{N_P(x)}{A} + \frac{M_P(x)}{I_y} z = -\left(\frac{P_H(x)}{A} + \frac{P_H(x)r(x)}{I_y} z\right) \tag{7}$$

The horizontal component of the prestressing force, P_H, can be ascertained through analysis of the tendon geometry. The vector representing the prestressing force, $\boldsymbol{P}$, exhibits a dependency on the variable x. Nonetheless, given that prestressing losses are not accounted for in this analysis, it is assumed that the magnitude of the prestressing force remains invariant along the entire length of the beam, thus $|\boldsymbol{P}(x)| = P$. As illustrated in Figure 1, the prestressing force aligns with the direction of $\boldsymbol{r}'$. Consequently, the horizontal component of the prestressing force can be deduced by considering its angle α relative to the vector of the prestressing force:

$$P_H(x) = P \ \cos\left(\alpha(x)\right) = \frac{P}{\sqrt{1+r'^2}} \tag{8}$$

In the calculation of σ_P, the component attributable to normal force-induced stresses will be disregarded, as it solely contributes to axial deformations of the component without inducing deflection. Consequently, the stress arising from prestressing is postulated as follows:

$$\sigma_P(x,z) = \frac{M_P}{I_y} z = -\frac{Pr}{\sqrt{1+r'^2} I_y} z \tag{9}$$

Therefore, the strain energy can be expressed as follows:

$$\Pi_i = \frac{1}{2E}\int_V (\sigma_E + \sigma_p)^2 \, dV = \frac{1}{2E}\int_V \left(\frac{M_E}{I_y} z - \frac{Pr}{\sqrt{1+r'^2} I_y} z\right)^2 dV = \frac{1}{2E}\int_V \left(\frac{M_E}{I_y} z - \frac{M_P}{I_y} z\right)^2 dV \tag{10}$$

Evaluating the volume integral yields:

$$\Pi_i = \frac{1}{2E}\int_V \left(M_E - \frac{Pr}{\sqrt{1+r'^2}}\right)^2 \frac{z^2}{I_y^2} \, dV \tag{11}$$

$$= \frac{1}{2E}\int_0^L \int_{-\frac{B}{2}}^{\frac{B}{2}} \int_{-\frac{H}{2}}^{\frac{H}{2}} \left(M_E - \frac{Pr}{\sqrt{1+r'^2}}\right)^2 \frac{z^2}{I_y^2} \, dz \, dy \, dx \tag{12}$$

$$= \frac{BH}{2EI}\int_0^L \left(M_E - \frac{Pr}{\sqrt{1+r'^2}}\right)^2 dx \tag{13}$$

Given that the constants I_y, H, and B exert no impact on the optimization of this functional within the scope of this paper, especially in the specific scenario of prestressed beams with uniform cross-section, these constants will be excluded. Consequently, a new functional, designated as $\mathcal{J}_p$, will be considered for the optimization problem:

$$\min \mathcal{J}_p = \int_0^L (M_E + M_P)^2 \ dx = \int_0^L \overbrace{\left(M_E - \frac{Pr}{\sqrt{1+r'^2}}\right)^2}^{\mathcal{H}} dx \tag{14}$$

Now, applying the EULER-LAGRANGE equation [4] in order to find the optimal tendon layout:

$$\frac{\partial \mathcal{H}}{\partial r} - \frac{d}{dx}\frac{\partial \mathcal{H}}{\partial r'} = 0 \tag{15}$$

$$\frac{\partial}{\partial r}\left(M_E - \frac{Pr}{\sqrt{1+r'^2}}\right)^2 - \frac{d}{dx}\frac{\partial}{\partial r'}\left(M_E - \frac{Pr}{\sqrt{1+r'^2}}\right)^2 = 0 \tag{16}$$

$$\frac{\partial}{\partial r}\left(M_E^2 - 2M_E\frac{Pr}{\sqrt{1+r'^2}} + \frac{(Pr)^2}{1+r'^2}\right) - \frac{d}{dx}\frac{\partial}{\partial r'}\left(M_E^2 - 2M_E\frac{Pr}{\sqrt{1+r'^2}} + \frac{(Pr)^2}{1+r'^2}\right) = 0 \tag{17}$$

$$\frac{\partial}{\partial r}\left(-2M_E\frac{Pr}{\sqrt{1+r'^2}} + \frac{(Pr)^2}{1+r'^2}\right) - \frac{d}{dx}\frac{\partial}{\partial r'}\left(-2M_E\frac{Pr}{\sqrt{1+r'^2}} + \frac{(Pr)^2}{1+r'^2}\right) = 0 \tag{18}$$

$$-2M_E\frac{P}{\sqrt{1+r'^2}} + \frac{2P^2r}{1+r'^2} - \frac{d}{dx}\left(2M_E\frac{Prr'}{(1+r'^2)^{3/2}} - 2\frac{(Pr)^2r'}{(1+r'^2)^2}\right) = 0 \tag{19}$$

$$\begin{aligned}-2M_E\frac{P}{\sqrt{1+r'^2}} + \frac{2P^2r}{1+r'^2} - 2P\frac{\frac{d}{dx}\left(M_Err'\right)\left(1+r'^2\right)^{3/2} - M_Err'\frac{d}{dx}\left(1+r'^2\right)^{3/2}}{\left(1+r'^2\right)^3} \\ -2P^2\frac{\frac{d}{dx}\left(r^2r'\right)\left(1+r'^2\right)^2 - \left(r^2r'\right)\frac{d}{dx}\left(1+r'^2\right)^2}{\left(1+r'^2\right)^4} = 0\end{aligned} \tag{20}$$

$$\begin{aligned}-\frac{M_E}{\sqrt{1+r'^2}} + \frac{Pr}{1+r'^2} - \frac{\frac{d}{dx}\left(M_Err'\right)\left(1+r'^2\right)^{3/2} - M_Err'\frac{d}{dx}\left(1+r'^2\right)^{3/2}}{\left(1+r'^2\right)^3} \\ -P\frac{\frac{d}{dx}\left(r^2r'\right)\left(1+r'^2\right)^2 - \left(r^2r'\right)\frac{d}{dx}\left(1+r'^2\right)^2}{\left(1+r'^2\right)^4} = 0\end{aligned} \tag{21}$$

$$\begin{aligned}-\frac{M_E}{\sqrt{1+r'^2}} + \frac{Pr}{1+r'^2} \\ -\frac{\left((M_E)'rr' + M_E\frac{d}{dx}\left(rr'\right)\right)\left(1+r'^2\right)^{3/2} - 3M_Err'^2r''\sqrt{1+r'^2}}{\left(1+r'^2\right)^3} \\ -P\frac{\left(2rr'^2 + r^2r''\right)\left(1+r'^2\right)^2 - 4r^2r'^2r''^2\left(1+r'^2\right)}{\left(1+r'^2\right)^4} = 0\end{aligned} \tag{22}$$

$$\begin{aligned}-\frac{M_E}{\sqrt{1+r'^2}} + \frac{Pr}{1+r'^2} \\ -\frac{\left((M_E)'rr' + M_E\left(r'^2 + rr''\right)\right)\left(1+r'^2\right)^{3/2} - 3M_Err'^2r''\sqrt{1+r'^2}}{\left(1+r'^2\right)^3} \\ -P\frac{\left(2rr'^2 + r^2r''\right)\left(1+r'^2\right)^2 - 4r^2r'^2r''^2\left(1+r'^2\right)}{\left(1+r'^2\right)^4} = 0\end{aligned} \tag{23}$$

$$\begin{aligned}-M_E\left(1+r'^2\right)^{(7/2)} + Pr\left(1+r'^2\right)^3 \\ -\left(\left((M_E)'rr' + M_E\left(r'^2 + rr''\right)\right)\left(1+r'^2\right)^{3/2} - 3M_Err'^2r''\sqrt{1+r'^2}\right)\left(1+r'^2\right) \\ -P\left(\left(2rr'^2 + r^2r''\right)\left(1+r'^2\right)^2 - 4r^2r'^2r''^2\left(1+r'^2\right)\right) = 0\end{aligned} \tag{24}$$

$$\begin{aligned}-M_E\left(1+r'^2\right)^{(5/2)} + Pr\left(1+r'^2\right)^2 \\ -\left(\left((M_E)'rr' + M_E\left(r'^2 + rr''\right)\right)\left(1+r'^2\right)^{3/2} - 3M_Err'^2r''\sqrt{1+r'^2}\right) \\ -P\left(\left(2rr'^2 + r^2r''\right)\left(1+r'^2\right) - 4r^2r'^2r''^2\right) = 0\end{aligned} \tag{25}$$

With this highly nonlinear differential equation, it becomes evident that an analytical solution is not feasible. This is why, in previously published works [1, 2], a numerical solution was sought.

1.2. Simplified solution

As mentioned before, a major challenge with Equation 14 is its resistance to being solved analytically. However, using the slender beam approximation ($r'^2 \approx 0$), this equation can be solved through variational calculus methods. As a result, the functional $\mathcal{J}_p$ is simplified to the following expression:

$$\mathcal{J}_p = \frac{B}{2EI} \int_0^L \overbrace{M_E^2 - 2M_E P r + P^2 r^2}^{\mathcal{H}} \, dx \tag{26}$$

Now, applying the EULER-LAGRANGE equation:

$$\frac{\partial \mathcal{H}}{\partial r} - \frac{d}{dx}\frac{\partial \mathcal{H}}{\partial r'} = 0 \tag{27}$$

$$\frac{\partial}{\partial r}\left(M_E^2 - 2M_E P r + P^2 r^2\right) - \frac{d}{dx}\frac{\partial}{\partial r'}\left(M_E^2 - 2M_E P r + P^2 r^2\right) = 0 \tag{28}$$

$$2M_E P + 2P^2 r = 0 \tag{29}$$

$$M_E + Pr = 0 \tag{30}$$

$$\Rightarrow r = \frac{M_E}{P} \tag{31}$$

This is the known solution for prestressed slender beams, as noted by [5]. This solution also provides useful insights for the proposed functional. However, it is important to understand that this solution is an approximation and might not be accurate for more complex shapes. The critical point is that this provides a clear indication that the variational formulation of the problem of minimizing strain energy is correct. However, this can only be demonstrated using numerical methods. This has already been accomplished in previous publications [1, 2], thus, with this paper, the process from problem formulation to its resolution is completed.

2. Conclusions

In this paper, a potential analytical solution to the problem of tendon layout optimization, previously addressed in earlier research, was investigated. In these publications, the analytical solution was omitted due to its high complexity, yet without detailed examination. This paper explores the solution to the variational problem by delving into the EULER-LAGRANGE equation. The result is a clearly defined, highly nonlinear differential equation, which precludes the possibility of an analytical solution. This explains why earlier research favored a numerical approach. It is noteworthy that under the simplification of a slender beam, the EULER-LAGRANGE equation can be easily solved, resulting in a formula known in engineering practice, namely, the acting bending moment divided by the prestressing force. However, this formula is inadequate for more complex geometries or when greater precision in tendon layout is needed. Finally, this paper concludes that the path to a numerical solution for the formulated problem of tendon layout optimization is definitive.

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