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Design of gridshells consisting of planar curves using Laguerre geometry

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Abstract

Owing to advancements in analysis and construction techniques, an increasing number of gridshells are designed and constructed for covering large space. Planarity of piecewise linear grid curves formed by beam elements leads to high constructability and cross-sectional compatibility at joints. However, a stiff tension ring is needed for transmitting the external loads to the supporting structure mainly through axial forces of members, constraining interior space and design freedom. This study proposes a design method for gridshells using Laguerre geometry, specifically the L-isothermic L-minimal surface. Its planar curvature lines and adjustable principal stresses in the principal curvature direction offer advantages in constructability and mechanical properties. By placing the beams along the curvature lines of this curved surface to design a gridshell, it is expected to support loads primarily through axial force. We propose a method to adjust stress distribution along the curvature lines of the continuous shell so that the stress approaches zero at a specific point on the surface boundary under pressure load. Thus, optimizing beam cross-sections of the gridshell to approximate stress distribution of the L-isothermic L-minimal surface enables designing gridshells without strong edge beams along one edge.

Keywords: gridshell, Laguerre geometry, canal surface, form finding, cross-sectional optimization

1. Introduction

Advancements in architectural technology have led to the development of various methods for designing curved structures covering large spaces with increased design flexibility. For example, the geometry of curves and surfaces [1], evolving from traditional spherical/cylindrical shells and regular truss structures, can be optimized to efficiently distribute internal forces. This approach has historical roots dating back to the use of arches and vaults. Modern developments include free-form shell structures, widely embraced in contemporary architecture for their ability to cover large spaces. There are many researches focusing on harmonizing structural integrity and creativity in architectural design employing curved surfaces.

A notable development for curved latticed shells is the gridshell, where structural members are arranged in a grid pattern. For instance, the concourse at King's Cross Station in London features a gridshell roof, creating a visually striking and expansive space. A mesh on a curved surface represented by a gridshell is constructed using triangular or quadrilateral elements, employing geometric properties defined by differential geometry [2]. The surface shape and the cross-section of each member in a gridshell are often optimized to realize a desirable internal force distribution where the axial force is dominant in each member and to minimize the shear and torsion of members and grid units. Additionally, we can reduce construction costs and enhance the stiffness and stability of the structure by arranging joints along planar curves [3].

The membrane theory addresses the equilibrium of a shell structure dominated by in-plane tension, compression, and shear forces, and accordingly, bending and torsional moments as well as the out-of-plane shear forces are neglected. Surfaces with specific force distributions under the uniform pressure load, such as isothermic [4] and membrane O surfaces [5], have been studied in the field of differential geometry. Furthermore, the application of Laguerre geometry [6] has also been studied in relation to discrete differential geometry. Laguerre geometry particularly offers mechanically efficient surfaces with adjustable stress distributions under the uniform pressure load, and this characteristic can be utilized in the design of grid shells along surfaces defined by the Laguerre geometry. However, those studies in applied mathematics consider only the equilibrium of forces and neglect material properties and compatibility of strains and deformation. Therefore, the actual force distribution in the deformed structure under the specified loads is unknown.

In this study, we propose a design method for a gridshell using the L-minimal generalized Dupin cyclide, which is an L-isothermic L-minimal surface. This surface, formulated by Schief et al. [7], offers high constructability due to the planarity of curvature lines, allowing for efficient arrangement of structural members of a gridshell. It also exhibits a favorable membrane stress distribution where the directions of principal stresses coincide with the directions of principal curvatures under the uniform pressure load. Therefore the load may be supported primarily through the axial forces of beam members in a gridshell. The main load applied to the gridshell is the vertical gravity load, but in this paper the normal load is applied. Although vertical and normal loads are inherently different, the direction and magnitude of vertical loads can be approximated by the normal load for shallow shells. Furthermore, it is demonstrated that there is one parameter in the stress distribution for normal uniform loading when deformation is not considered, and the distribution is not uniquely determined [7]. Our study confirms the existence of a stress distributions along curvature lines where a principal stress in one direction becomes zero at a certain point by appropriately assigning the value of the single parameter. We then generate an axial force distribution in the beams of a gridshell to achieve zero axial force at a member, aligning the members along the curvature lines of the shell surface. Next, we design a gridshell through cross-sectional optimization to minimize the norm of reaction forces along the boundary to ensure that both geometric and mechanical properties are maintained. The actual internal force distribution when the elastic deformation of the members is considered is confirmed by structural analysis to verify the preservation of favorable mechanical properties of the gridshell. Although mechanical methods for minimizing reaction forces at the edge, such as prestressing [8], have already been demonstrated, our study uses a geometric method.

2. Equilibrium on a curved surface

In this section, we summarize the stress distribution of an L-minimal generalized Dupin cyclide for completeness of the paper. The surface is classified as an L-isothermic canal surface. By utilizing the properties of the surface, we can determine the tensile and compressive forces that equilibrates to the uniform normal loads.

2.1. L-minimal generalized Dupin cyclide

The geometry of L-minimal generalized Dupin cyclide [7] is represented using parameters corresponding to curvature line coordinates, denoted by α and β. Introducing $u = \arccos(\frac{1}{\cosh \beta})$ as a parameter, and expressing the surface in terms of α and u, the expression for the surface is as follows:

$$\mathbf{r} = -\frac{\cos\alpha}{2(a_0 - c_0\cos\alpha\cos u)}\begin{pmatrix}\cos\alpha \\ a_0\sin\alpha \\ \cos\alpha\sin u\end{pmatrix} + \frac{1}{4}\begin{pmatrix}-a_0 \\ 2\alpha \\ 0\end{pmatrix} \tag{1}$$

where a_0 and c_0 are constants.

This surface is obtained as the envelope of a sphere whose center moves along a cycloid. The characteristics of this surface are as follows:

1. The curvature lines are segments of circles and therefore planar curves.
2. Principal stresses occur in the direction of curvature lines in response to normal direction loads.
3. The distribution of the principal stresses for applied normal loads is defined with a single free constant.

These characteristics correspond to the following advantages when adopting the surface for designing a gridshell in architectural structures:

1. The assemblage of the beam members can be done on a flat surface, which makes construction easier.
2. Since the axial forces of the members mainly resist the load, the strength of the member can be used efficiently.
3. A mechanically efficient structure can be created by adjusting the distribution of axial forces.

Figure 1: L-minimal generalized Dupin cyclide

2.2. Shell membrane theory

In the shell theory, assuming zero out-of-plane moment and shear allows us to obtain the equilibrium equations of the membrane theory.

The surface used in this study is a membrane O-surface where in-plane shear force does not exist in the principal curvature directions. Consequently, the equilibrium equation between the constant normal load on the surface whose magnitude per unit area is denoted by Z and the stresses per unit length denoted by T_1 and T_2 in the principal directions corresponding to the curvature line coordinates α and

β, respectively, can be expressed as follows:

$$
\begin{aligned}
&T_{1\alpha} + (\ln A_2)_\alpha (T_1 - T_2) = 0 \\
&T_{2\beta} + (\ln A_1)_\beta (T_2 - T_1) = 0 \\
&\kappa_1 T_1 + \kappa_2 T_2 + Z = 0
\end{aligned}
\tag{2}
$$

The subscripts α and β in Eq. (2) represent the partial differentiation with respect to α and β, respectively. Additionally, $A_1 = ||\mathbf{r}_\alpha||$ and $A_2 = ||\mathbf{r}_\beta||$, and κ_1 and κ_2 are the principal curvatures in the α and β directions, respectively.

Schief et al. [7] derived the stress resultants on the L-isothermic canal surface as

$$
T_1 = -\frac{Z}{2\kappa_2} - \frac{I_0}{\kappa_2 A_2^2}, \quad T_2 = -\frac{Z}{2\kappa_1}\left(1 - \frac{P^2}{A_1^2}\right) + \frac{I_0}{\kappa_1 A_1^2} \tag{3}
$$

where $P = A_1 - A_2$, and I_0 is an arbitrary constant. By varying I_0, various equilibrium states can be obtained. In addition, by substituting 0 for Z, we confirmed that the term containing I_0 corresponds to the stress in the self-stress state satifying the thirf equation in Eq.(2).

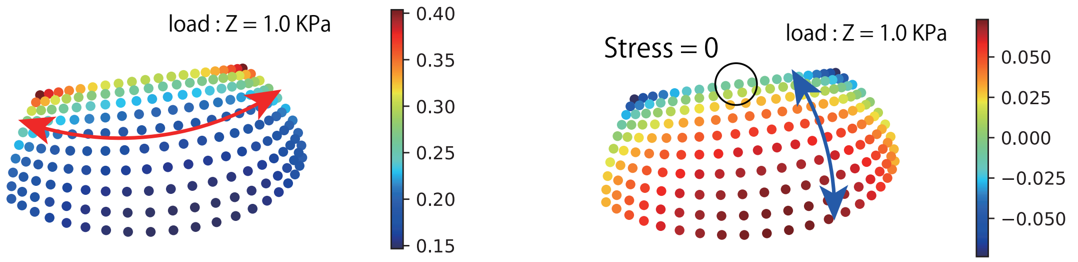
2.3. Adjustment of stress distribution

By assigning a value for I_0 in Eq. (3), a stress distribution can be obtained. Therefore, it is possible to obtain a specific I_0 that gives zero stress at a certain point.

$$
T_1 = 0 \Leftrightarrow I_0 = -\frac{1}{2} A_2^2 Z \tag{4}
$$

$$
T_2 = 0 \Leftrightarrow I_0 = \frac{1}{2}(A_1^2 - P^2) Z \tag{5}
$$

In the following examples, the surface shown in Fig. 2 is considered. The region with four edge curves is extracted from the surface, and I_0 is calculated to ensure zero stress at the midpoint of the upper edge in the direction corresponding to T_2 specified in the blue arrow in Fig. 2(b) . The resulting stress distribution is shown in Fig. 2.



(a) T_1 (b) T_2

Figure 2: Stress distribution with zero longitudinal stress at the center of the top edge ($\times 10^3$ N/m)

2.4. Discretization of surfaces

To design a gridshell, the surface is discretized into quadrilateral mesh. As shown in Fig. 3, nodes are placed at the intersections of curvature lines drawn at constant intervals of α and u, and beams are arranged to connect these nodes. To compare the axial force distribution of the beams obtained from structural analysis with the stress distribution of the continuous shell, we integrate the stress of the shell over the interval of the grid to obtain the corresponding axial force distribution. Additionally, in the structural analysis, the distributed normal directional load applied to the surface is transformed to the concentrated loads at each node whose values are determined through the integration over the region covered by each node. This allows us to approximately correlate the distributed loads and stress distribution on the continuous surface to the concentrated loads at each nodes and axial force distribution of the gridshell.

3. Structural Optimization

The gridshell generated from the L-minimal generalized Dupin cyclide is supported along its perimeter. If the reaction forces along the upper edge in Fig. 3 approach zero, its force distribution corresponds to the vanishing axial forces of the members connected to the upper edge. Therefore, we minimize the norm of reaction forces at the upper edge through cross-sectional optimization and compare the resulting axial force distribution to the stress distribution of the shell. In this paper, we also consider the compatibility of deformation, which requires structural analysis and prevents us from directly specifying the beam cross-section based on the stress distribution.

3.1. Problem formulation and conditions

A hollow circular steel pipe is used for the beam section and its outer radius is determined by solving the following structural optimization problem:

$$\begin{aligned} &\text{find} && \mathbf{d} \\ &\text{minimize} && ||\mathbf{F}_{\text{edge}}||^2 \\ &\text{subject to} && 3 \leq d \leq 20 \end{aligned} \tag{6}$$

where $\mathbf{F}_{\text{edge}}$ represents the vector of reaction forces at the upper edge support points and $\mathbf{d}$ is a vector of the cross-sectional radii of the beams. The initial cross-section, the material properties, and the value of normal load are shown in Table 1. The L-BFGS-B method is used for optimization using the python library scipy.optimize.minimize [9].

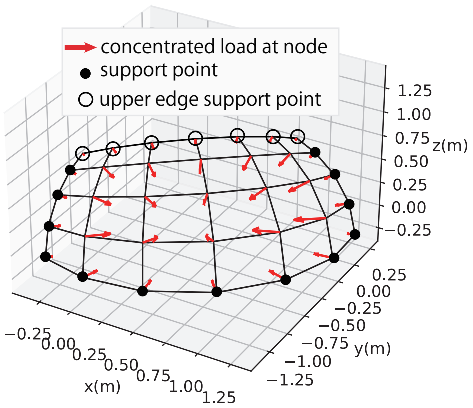


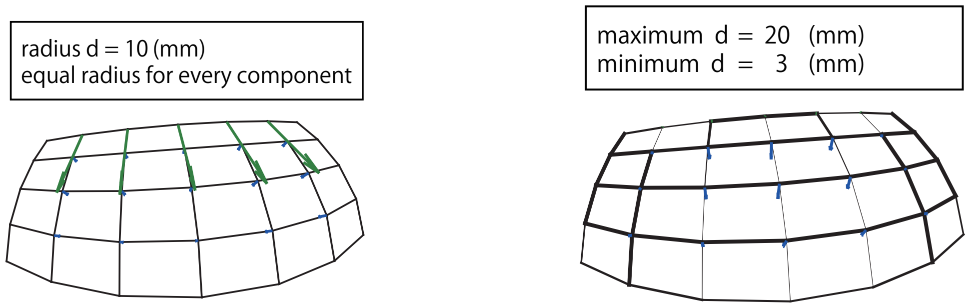
Figure 3: Load and boundary conditions

Parameter	Value
Initial radius d (mm)	10
Thickness t (mm)	1
Young's modulus E (GPa)	100
Poisson's ratio ν	0.3
Load per unit area Z (KPa)	1.0

Table 1: Conditions

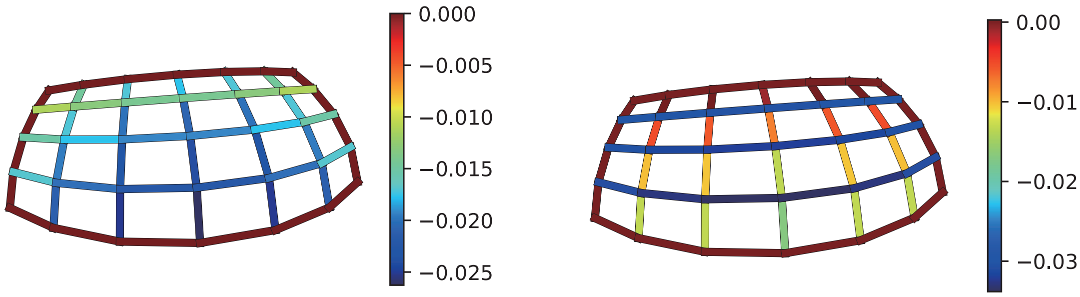
3.2. Results of optimization

The solution of the optimization problem results in nearly zero support reaction forces at the upper edge, as shown in Fig. 4. This result corresponds to the axial force distribution, where the axial force in the beam connected to the center of the upper edge becomes zero after optimization as shown in Fig. 5. Moreover, the maximum deformation is less than 1/200 of the span; therefore, no excessive deformation occurrs. Although details are not shown, shear forces and bending moments are negligible compared to the axial force. As a result, we have achieved a gridshell that predominantly withstands the load through axial force, thereby eliminating the necessity of stiff edge beams along the perimeter.



(a) Initial state (b) After optimization

Figure 4: Displacement (blue arrows), support reactions (green arrows); line thickness is proportional to beam radius (Arrows are displayed at 100 times larger for visibility.)

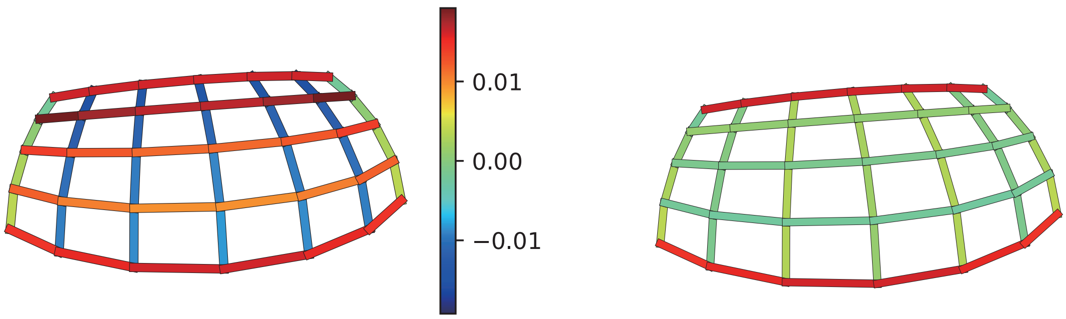


(a) Initial state (b) After optimization

Figure 5: Axial force distribution ($\times 10^3$N)

3.3. Comparison with the stress distribution of shell

The axial force distribution obtained through optimization is compared with the stress distribution of shell obtained under the assumption of zero stress at the center of the upper edge of the surface. In this comparison, the stress distribution in the shell is converted into an axial force distribution by integration, which is hereinafter referred to as the ideal axial force distribution. It is confirmed from Fig. 6 and Table 2 that the axial force distribution matches the stress distribution after optimization. Note that beams along the outer perimeter are pin-supported at both ends; therefore, their axial forces are indeterminate, leading to differences from the stress distribution.



(a) Initial state (b) After optimization

Figure 6: Difference between analytical solution and ideal axial force distribution ($\times 10^3$ N)

Table 2: Difference between analytical solution N, and ideal axial force $N_t (\times 10^3$ N)

	Initial state	After optimization
Maximum value of $N - N_t$	1.923×10^{-2}	3.126×10^{-3}
Minimum value of $N - N_t$	-1.539×10^{-2}	-2.474×10^{-3}
Average of $\|N - N_t\|$	1.239×10^{-2}	2.474×10^{-3}

4. Conclusion

In this study, we proposed a design methodology for gridshell structures using L-isothermic L-minimal surfaces. The gridshell designed using the proposed method resists primarily by axial forces against normal loads. Previous studies [7] have shown that the stress distribution in L-isothermic L-minimal surfaces can be expressed in terms of a single arbitrary constant, and using this property we have shown that the constant can be determined so that the principal stress in a single direction vanishes at a certain point. The continuous shell is converted to a gridshell, and structural analysis is carried out to find the actual axial force under normal loads. An optimization problem is formulated to minimize reaction forces at the specified edge of the gridshell with the cross-sectional radii of the beam members as design variables. The optimization result verifies that the gridshell can be designed so that the axial force vanishes at a point along the perimeter even when deformation corresponding to the actual material property is considered. Thus, a design without a stiff edge beam or reaction from the supporting structure can be obtained.

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