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Constructing Topological Interlocking Assemblies Based on an Aperiodic Monotile

Reymond Akpanya*, Tom Goertzen[a], Yuanpeng Liu[b], Sascha Stüttgen[c], Daniel Robertz[c], Yi Min Xie[b], Alice C. Niemeyer[a]

* Chair of Algebra and Representation Theory, RWTH Aachen University
Pontdriesch 10-12, 52062 Aachen, Germany
reymond.akpanya@rwth-aachen.de

[a]Chair of Algebra and Representation Theory, RWTH Aachen University, Aachen, Germany
[b]Centre for Innovative Structures and Materials, School of Engineering, RMIT University, Melbourne, Australia
[c]Chair of Algebra and Number Theory, RWTH Aachen University, Aachen, Germany

Abstract

The einstein problem asks whether there exists an aperiodic monotile, i.e. a tile whose copies can be only arranged as a nonperiodic tessellation of the Euclidean plane. In 2023, Smith et. al. proved the existence of such a tile by constructing a tile, called “the hat”, and described the corresponding tiling of the plane. Moreover, they describe a two-parameter family $\mathrm{Tile}(a, b)$ of such monotiles, where $\mathrm{Tile}(1, \sqrt{3})$ is the hat. Modifying the edges of $\mathrm{Tile}(1, 1)$ in a certain way leads to a chiral aperiodic monotile, which is part of a family called “spectres”. Based on this construction, we propose blocks that can be arranged into infinite topological interlocking assemblies obeying the same assembly rule as the spectre tessellation. A topological interlocking assembly is an assembly of blocks that is constrained by a fixed frame such that each set of blocks is kinematically constrained. Our construction establishes the existence of aperiodic topological interlocking assemblies. In this paper, we present details of the proof that our constructed assemblies are indeed topological interlocking assemblies. We highlight the combinatoric and geometric properties of our constructed blocks and discuss possibilities of constructing our topological interlocking assemblies without causing any collision of blocks during assembly.

Keywords: Topological interlocking, 3D-printing, computational form finding, aperiodic monotile, einstein problem, Hangai Prize applicant

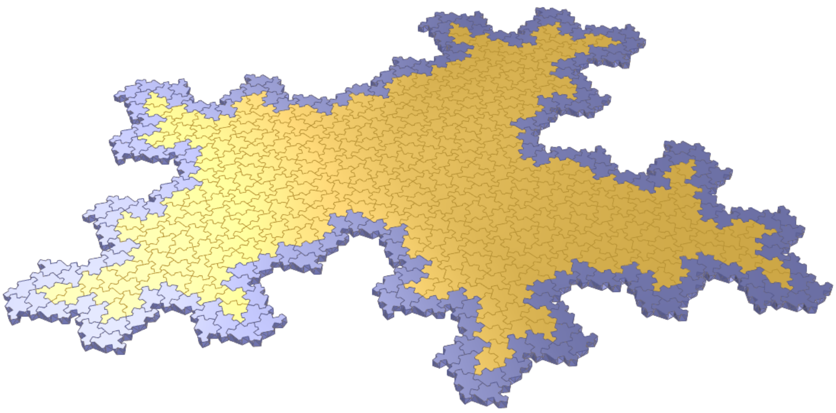


Figure 1: Supercluster of interlocking blocks with peripheral frame composed by most outer blocks coloured in grey

1. Introduction

Modular construction strategies are increasingly seen as a promising way to address upcoming challenges faced by the construction sector while also offering the potential to optimise sustainability. A fruitful approach for tackling this optimisation challenge is the utilisation of topological interlocking assemblies (TIA), which are assemblies of blocks each constrained from moving in any direction by its neighbouring blocks and a peripheral frame. The extensive study of TIA was initiated in [1] by the mechanical investigation of a TIA based on tetrahedra. Over the years, various methods that produce different types of TIA have been established. These construction methods often originate from the idea that a tessellation grows out of the plane to obtain an assembly of blocks [2]. For instance, blocks that generate TIA include convex blocks [3], blocks based on Voronoi domains [4, 5] and blocks that are constructed by exploiting an Escher-like approach involving the deformation of non-convex fundamental domains [6, 7]. Assemblies constructed by these types of blocks are often symmetrical and based on a single block geometry. To this day, it is unknown whether a block exists that yields an aperiodic TIA.

In 2023, Smith et al. solved the long-standing einstein problem [8]. This problem asks whether a two dimensional shape that only admits an aperiodic tessellation of the Euclidean plane exists. The einstein problem is essentially an expansion of the second aspect of Hilbert's eighteenth problem which inquires about the existence of a singular polyhedron that can tile the Euclidean 3-space, yet without any of the tiling configurations being isohedral. Moreover, in [9] the authors establish the existence of a family of tiles, called spectres, where each tile only tiles the Euclidean plane aperiodically. While there are no symmetries respecting the entire tessellation, each tile of a given tessellation can be obtained solely through rotations and translations, excluding reflections of the tile.

Based on this work, we establish the existence of a 2-parametric family of three-dimensional interlocking blocks that can only be assembled in an aperiodic way. The idea of the construction is to interpolate between different tiles of the spectre-family in order to construct the desired interlocking blocks. Furthermore, we analyse the assemblability and interlocking property of the blocks that result from the above construction in the context of a parameter study. This is achieved by employing already established procedures to test whether a given assembly is a TIA [10, 11, 12]. In order to conduct our investigations we make use of the programming language Julia [13]. In particular, we exploit the implementations that are provided in our Julia package *Non-convex-interlocking* [14]. Additionally, we provide STL-files that contain the blocks that we have constructed in our investigations, see [15].

2. Description of our proposed spectre-block

Our proposed spectre-block B_p is based on the idea of interpolating between two spectres placed in two parallel planes, a base and a top plane. The 'base' spectre is the tile $\mathrm{Tile}(1,1)$, see Figure 2a, and the 'top' spectre controlled by a point $p = (p_1, p_2)^t$ is shown in Figure 3 for $p = (-0.25, 0.75)^t$. The main idea is to interpolate between the base and the top spectre such that every tile in between forms a spectre.

(a) (b)

Figure 2: (a) The base tile $\mathrm{Tile}(1,1)$, (b) the mystic, i.e. the union of two base tiles

In the following, we first describe the base and top tile for the block B_p for a given p and exploit the construction of these tiles to derive a triangulation of the boundary of B_p.

2.1. The base tile $\mathrm{Tile}(1, 1)$

We aim to define the base tile as a list V of 14 points in $\mathbb{R}^2$ that form the vertices of the boundary. We proceed as follows: First, we define the list I of the 14 inner clockwise angles of consecutive angles of the base tile: $I = [270, 120, 180, 120, 90, 240, 90, 240, 90, 120, 270, 120, 90, 120]$. Note, that the sum of all angles in I is given by $2160 = (14 - 2) \cdot 180$ since the base tile has exactly 14 vertices and 14 edges. In order to obtain the base tile, we need to define a rotation function $\mathrm{RotatePointClockwise}(q, c, \alpha)$ that rotates a point $(q_1, q_2)^t = q \in \mathbb{R}^2$ clockwise around a centre $c \in \mathbb{R}^2$ by an angle α. Next, we initialise $V = [(0, 1)^t, (0, 0)^t]$. Recursively, we define the $i{+}2$-th entry of V for $1 \leq i \leq 12$ as $V[i{+}2] := \mathrm{RotatePointClockwise}(V[i], V[i{+}1], I[i])$.

2.2. The top tile

The 28 vertices V' of the top tile are obtained by a similar procedure as described above. Let therefore $p = (p_1, p_2)$ be a control point and V' be initialised by $V' = [(0, 1), p]$. For $1 \leq i \leq 13$ we define the $2i$-th and $2i + 1$-th entries of the list V' as $V'[2i + 1] := V[i + 1]$ and $V'[2i + 2] := \mathrm{RotatePointClockwise}(V'[2i], V'[2i{+}1], I[i])$. For instance, consider the point $p = (-0.25, 0.75)^t$. The process of obtaining the top tile for this point is exemplified in Figure 3.

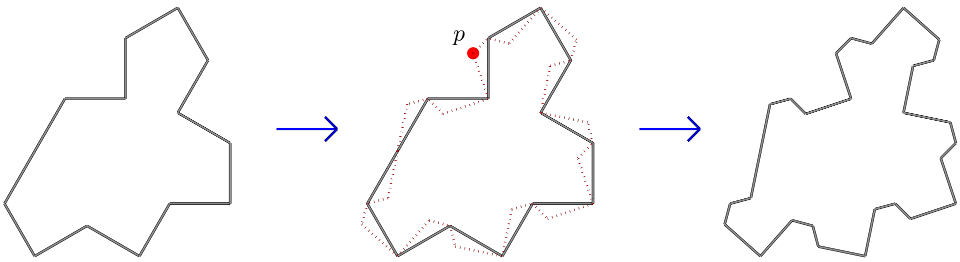


Figure 3: Modifying the edges of the base tile to obtain the spectre for $p = (-0.25, 0.75)$

Figure 4 illustrates different top tiles that result from the above construction.

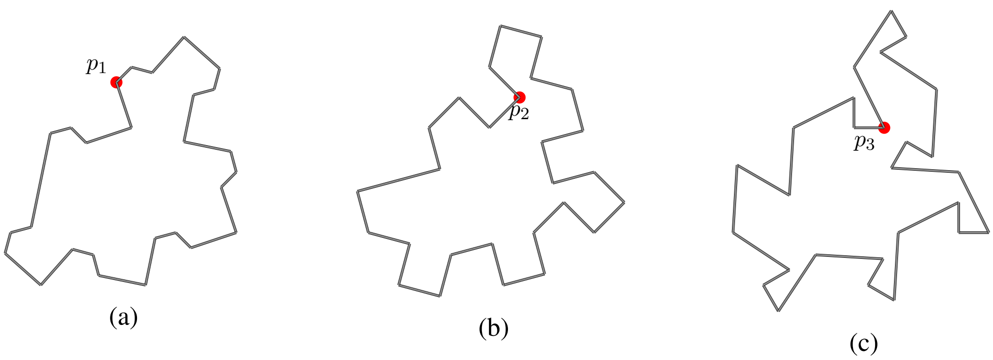


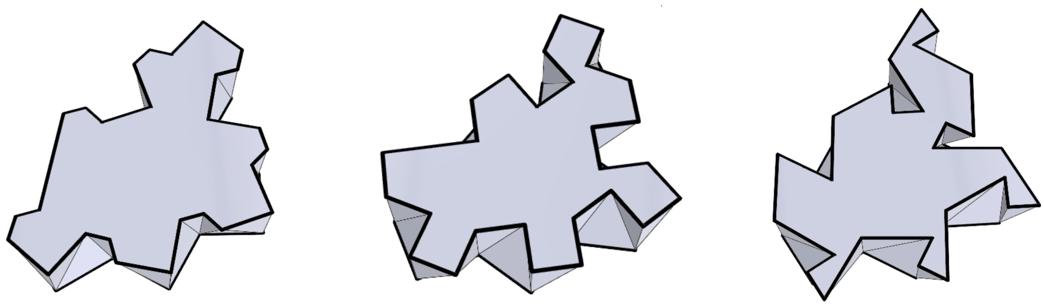
Figure 4: The spectre obtained by modifying the edges of the base tile using the point (a) $p_1 = (-0.25, 0.75)^t$, (b) $p_2 = (0.5, 0.5)^t$ and (c) $p_3 = (0.5, 0.0)^t$

2.3. Block construction – Interpolating between base and top tile

For the block coordinates, we place the two lists V and V' in two different parallel planes by embedding them into $\mathbb{R}^3$: vertices of the base tile V with their third coordinate set to 0; vertices of the top tile V' with their third coordinate set to 1. The faces of the block are then given as a list of faces (`faces`) based on the constructed points. For each $1 \leq i \leq 14$ we add three faces to `faces`, each consisting of three vertices. These faces form triangles representing the surfaces of the block.

- The first face includes the vertices $V[i]$, $V'[2i-1]$ and $V'[2i \mod 28+1]$.
- The second face includes the vertices $V[i]$, $V[i+1]$ and $V'[2i]$.
- The third face includes the vertices $V[i \mod 14+1], V'[2i]$ and $V'[2i \mod 28+1]$.

Finally, we add the faces for the bottom and top tile to `faces`. The triangulation is motivated by the Escher-like approach for constructing topological interlocking assemblies with wallpaper symmetries, even though the assemblies obtained in this paper do not posses wall-paper symmetries, see [6, 7].



(a) (b) (c)

Figure 5: Top view of the spectre block obtained from interpolating between base tile and the modified spectre using (a) $p = (-0.25, 0.75)^t$, (b) $p = (0.5, 0.5)^t$ and (c) $p = (0.5, 0.0)^t$

3. Interlocking property of the spectre-block

Next, we discuss different assemblies of the proposed spectre-block B_p. In particular, it is shown that the spectre-block admits an aperiodic TIA, i.e. a TIA consisting of infinitely many identical blocks that are arranged in a planar aperiodic manner. Furthermore, the methods to establish the interlocking property of a given assembly of blocks are introduced and exploited to discuss the interlocking property of finite assemblies of our proposed spectre-block.

3.1. Assemblies of the spectre-block

Here, a block is a 3-dimensional manifold with polyhedral boundary. An assembly of blocks is a placement of the blocks in Euclidean 3-space such that any two blocks may only intersect at their boundary. In [9], a method to generate various planar tilings consisting of a finite number of identical spectre tiles is described. In particular, Smith et al. formulate a substitution rule that can be applied to a tiling of spectres to create a larger tiling. The construction that Smith et al. establish to generate finite tilings of the spectre, yields an unique aperiodic tiling of the Euclidean plane by the spectre. Here, we give an abbreviated description of this substitution rule. For a more detailed description, we refer to [9].

Starting from a given tiling consisting of the spectre $\mathrm{Tile}(1, 1)$, Smith et al. subdivide the tiling into spectres and mystics, i.e. a union of two spectres (see Figure 2b) and identify key points for each of these tiles. They then replace each spectre by a particular spectre cluster (see-Figure 6a) and each mystic by a specific mystic-cluster (see Figure 6b). By identifying key points of the different clusters such that the new key points are in a one-to-one correspondence with the key points of the spectres and mystics, Smith et al. are able to generate a tiling of spectre and mystic clusters such that key points of the clusters coincide if and only if the corresponding key points of the spectres and mystics coincide. This procedure can then be iterated to produce bigger tilings containing copies of the spectre.

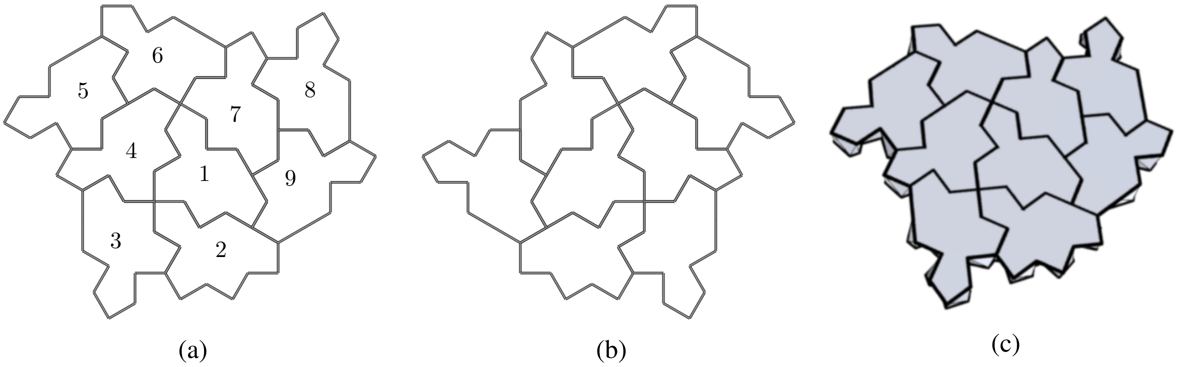


Figure 6: (a) The spectre cluster and (b) the mystic cluster, (c) Bottom view of an assembly of spectre-blocks that corresponds to the spectre cluster shown in (a)

Since our block is derived from interpolating between the spectre in one plane and a new tile that is obtained by deforming the spectre in a parallel plane, we can use the above substitution rules to create assemblies of our proposed spectre-block. For instance, Figure 6c shows the bottom view of an assembly that results from applying the described substitution rules to our spectre-block. Hence, the proposed spectre-block B_p can be used to construct a planar aperiodic assembly.

3.2. Interlocking property

As described in Section 1., a TIA is an arrangement of blocks together with a fixed frame such that no subsets of blocks can be removed from the assembly. Here, we give an abbreviated version of the precise definition of a topological interlocking assembly in [6].

Definition 1 *Given a family of blocks $(X_i)_{i\in I}$, where $X_i \subset \mathbb{R}^3$ is a compact subset and I is an index set, we say that $(X_i)_{i\in I}$ is a topological interlocking assembly for a given frame of fixed blocks $J \subset I$, if the following holds:*

1. *$(X_i)_{i\in I}$ is an assembly, that is any two blocks can only intersect at their boundaries;*
2. *Any finite set of blocks $\emptyset \neq S \subset I \setminus J$ cannot be moved using continuous motions fixing the frame and without causing intersections of the blocks (interlocking property).*

Thus, in order to establish the interlocking property of a given assembly of blocks, all continuous movements of all subsets of blocks have to be considered and analysed. Since this turns out to be a highly complex task, alternative methods have to be introduced to verify the interlocking property of a proposed assembly. One such method is described in [10]. Wang et al. translate the task of determining whether a given assembly forms a TIA into a linear program whose solvability gives insights into the interlocking property of the assembly. Instead of considering continuous movements, the authors consider infinitesimal movements and generate non-penetration inequalities of blocks of the assembly, i.e.

linear inequalities which enforce that any two blocks of the assembly that are in contact with each other cannot intersect each other after being moved in space with corresponding infinitesimal movements. Thus, Wang et al. state that the interlocking property holds, if there exists no non-zero solution of the described linear program. Note, Wang et. al formulate this interlocking test for assemblies consisting of convex blocks only. In [11] this method for the verification of the interlocking property is generalised for assemblies of non-convex blocks. By using this generalised interlocking test, we are able to verify that the assemblies presented in this paper are indeed topological interlocking assemblies.

4. Assemblability of the spectre-block

In disciplines such as civil engineering it is important that an assembly can actually be built block by block. Designing a motion plan for the blocks of the assembly is therefore essential to ensure that the entire assembly can be constructed without causing any intersections of the blocks. This careful planning guarantees a smooth construction process, minimising the risk of disruptions and ensuring structural integrity. Of particular interest is the investigation of the assemblability of a given assembly by restricting the possible motions of the different blocks of the assembly to in plane translational motions only. If the restriction of the space of possible motions of the different blocks yields an assembly plan, then this plan facilitates an easy way to actually construct the given assembly from the corresponding building blocks on site. Here, we address the fact that the interlocking tests described in [10, 11] can be used to gain first insights into the design of a possible motion plan for a given assembly. In order to investigate the design of a motion plan, we recursively define the *assemblability* of a given assembly as follows:

Definition 2 *Let I be an index set and $(X_i)_{i\in I}$ be an assembly of blocks. We say that the assembly $(X_i)_{i\in I}$ is assemblable, if it consists of only one block or it is the union of two assemblable subassemblies $A = (X_i)_{i\in J}$ and $B = (X_i)_{i\in K}$ where J and K are non-empty sets satisfying $J \cup K = I$, such that there exist simultaneous motions of the blocks in A, that separate the assembly A from B without causing any penetration of blocks.*

Thus, if a given assembly is assemblable, it is possible to disassemble it in a sequential manner, until all blocks are separated from each other. Reversing this process gives rise to a sequence of assembly steps that need to be performed in order to build the assembly from a collection of individual blocks. Hence, a desired motion plan of the block is derived. Note that the second condition of the above definition states that the assembly $(X_i)_{i\in I}$ with the blocks in B defined as a corresponding frame does not yield a TIA. So, the interlocking tests described in Section 3.2. can be exploited to examine the assemblability of a given assembly. Note that these tests formulate sufficient conditions for an assembly to be topologically interlocked and can therefore not prove that the given assembly is not a topological interlocking assembly. However, when restricting Definition 1 and Definition 2 to only allow translational motions as possible motions of the blocks, the existence of a finite motion not causing penetration of the blocks is equivalent to the existence of an infinitesimal one. This is due to the formulation of the linear constraints considered by the mentioned tests.

5. Parametric study on assemblability of the spectre-block

Here, we investigate the different properties of our constructed blocks depending on the choice of the parameters that are used to obtain the given blocks. In particular, we examine the constructed blocks with respect to the interlocking property of the assemblies when applying different restrictions to the possible motions and assemblability of the whole assembly. These observations are conducted by analysing the assemblies of the different blocks that correspond to the spectre cluster, see Figure 6a.

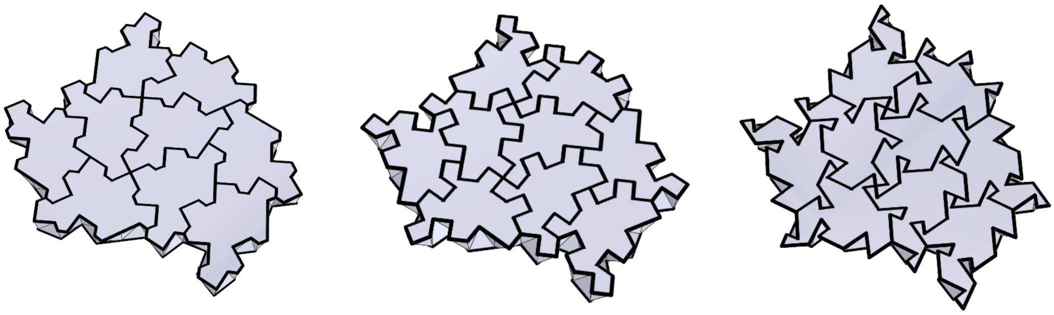


Figure 7: Top view of the assemblies of spectre-blocks B_p that are obtained from interpolating between base tile and the modified spectre using the parameterising points (a) $p = (-0.25, 0.75)^t$, (b) $p = (0.5, 0.5)^t$ and (c) $p = (0.5, 0.0)^t$

For the interlocking tests, we consider the outer blocks as frame and check for the existence of infinitesimal motions of the inner block. For the assemblability tests we apply the interlocking test to a recursive sequence of sub-assemblies. Since it would not be feasible to test all possible assembly sequences, we restrict ourselves to test only promising sequence candidates, that we have determined by experimenting with 3D-printed models of the blocks, see Figure 8.

(a) (b)

Figure 8: Various examples of assemblies of 3D-printed spectre-blocks

By pursuing this approach, we are able to find a suitable sequence for the block with $p = (-0.25, 0.75)^t$. In order to give the desired assembly plan we order the 9 blocks of the assembly corresponding to the ordering of the spectre tiles in Figure 6a. It can be assembled in the following way:

- We start constructing the assembly by placing the block 6;
- we add block 5 to the assembly;
- we add block 4 to the assembly;
- we add the sub-assembly consisting of the blocks 1 and 7 to the assembly;
- we add sub-assembly consisting of the blocks 8 and 9 to the assembly;
- we add sub-assembly consisting of the blocks 2 and 3 to the assembly.

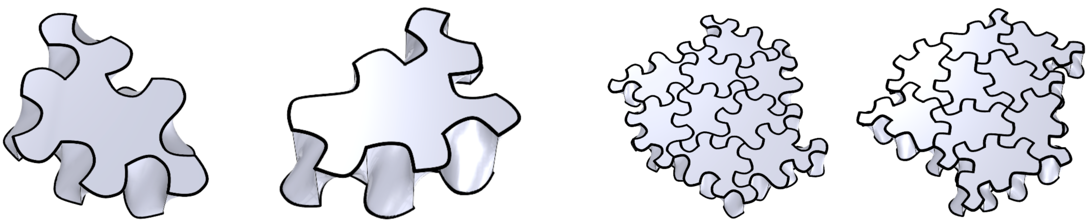
Again by experimenting with 3D-printed blocks to propose assembly sequences, we test the assemblability of the assemblies arising from different choices of parameters. We summarise our results in Table 1. Note that a "yes" in column "Assemblability" means that the interlocking test could not verify the interlocking property. It does not mean that we are able to prove the existence of a finite continuous motion of the blocks that does not cause penetration. In the column "Assemb. allowing only in plane transl. motions" we restrict the assemblability definition, and thus the allowed motions for topological interlocking during the test, to translational motions of the blocks parallel to the xy-plane. Thus a "yes" in this column actually corresponds to the existence of an assembly sequence with finite motions.

Table 1: Results of interlocking property and assemblability tests depending on the choice of parameters. The assemblability tests have been performed on derived assembly sequences that arise from experiments with 3D-printed blocks.

Parameter (a, b)	Int. Property	Assemblability	Assemb. allowing only in plane transl. motions
$(-0.5, 1.0)$	Yes	No	No
$(-0.25, 0.75)$	Yes	Yes	No
$(0.25, 0.25)$	Yes	Yes	No
$(0.25, 0.0)$	Yes	No	No
$(0.5, 0.0)$	Yes	No	No
$(-0.5, 0.5)$	Yes	No	No
$(0.5, 0.5)$	Yes	No	No
$(-0.7, 0.8)$	Yes	No	No
$(0.6, 0.6)$	Yes	No	No

6. Curved spectre-block

As stated in [9], one can replace each edge of the base tile with a curve to create a new tile that admits a strictly chiral aperiodic tessellation. Based on this insight, we develop a method to construct a tessellation of 3D blocks with curved interfaces.



(a) (b) (c)

Figure 9: (a,b) Top and bottom view of the spectre-block that is obtained from interpolating between curved spectres (c) Top and bottom view of assemblies of curved spectre-blocks

This method begins with two layers of congruent tessellations of the base tile, separated by a certain distance of translation in the plane's normal direction. We then modify each layer by substituting its straight edge with a curve, whose shape can be almost arbitrary as long as no self-intersection appears within the tile. Next, for each pair of corresponding curves between the two layers, we generate a series of curves that smoothly transition between them. These curves serve as the skeleton, which can guide the creation of a curved transition surface passing through them. For each pair of corresponding tiles, we combine them with the transition surfaces corresponding to each edge to create a single block. By repeating this process for all pairs of tiles, we ultimately create a tessellation of 3D blocks with unique free-form interfaces.

When experimenting with 3D-printed versions of this block, it turns out that the resulting assemblies exhibit the geometrical interlocking property, i.e. no pair of adjacent blocks can be disassembled without causing collisions. That means, in order to assemble those blocks, one needs to exploit deformations in the construction material.

7. Conclusion

In this paper, we establish the existence of topological interlocking blocks that only admit aperiodic TIA. We investigate the interlocking property and the assemblability of the proposed TI blocks for certain parameters and further propose curved versions of these interlocking blocks that facilitate the same aperiodic TIA. Some of the blocks that are presented in this work cannot be assembled without causing intersections. By combining the block geometry with soft-materials such as rubber, it is still possible to assemble the blocks and guarantee a geometric interlocking property with the contact to other neighbouring blocks. In further research, we aim to examine larger assemblies of the proposed blocks. For instance, the question, whether the assemblies that correspond to supercluster can be assembled and also form TIA, arises. Moreover, one has to establish an analytical relationship between the parameters of a block and the interlocking property and assemblability of its corresponding clusters. To do so, it is necessary to investigate the assemblability of the different assemblies by means of explosion, i.e. simultaneous radially outwards motions of all blocks in an assembly. Since the presented results are purely computational, the mechanical behaviour and the manufacturing of the above blocks has to be addressed in further research.

8. Acknowledgements

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