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Exploring the potential of adaptive Monte Carlo simulation with importance sampling to predict the structural reliability of highly nonlinear tension structures

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Abstract

There is a desire to to create a Eurocode part that describes the design principles, rules, guidelines, and practices for tensioned membrane structures. This will have a significant impact on the design and construction of tensioned membrane structures, for engineering consultants and contractors, for whom a design code has not previously existed. The first stage in this process, the publishing of a technical specification [1], is approaching completion. Encompassing cable and membrane structure design within a Eurocode framework, at first sight, might not be expected to generate significant complications. Perhaps not immediately obvious, is that the calibration of the partial factors is undertaken within the context of the overarching EC0, where structural safety is defined in terms of a minimum probability of failure (or safety index) rather than a direct measure of a physical engineering quantity (e.g., stress). There is, therefore, a requirement to calculate (or estimate) the probability of failure for tensioned membrane structures. For a structural system such as a tensioned membrane structure that possesses geometrically nonlinear behaviour, and is constructed from nonlinear elastic and nonlinear plastic composite materials, calculating the probability of failure is a major challenge. This paper outlines some of the challenges, and explores the potential use of adaptive importance sampling Monte Carlo simulation to overcome (some of) them.

Keywords: Membrane structures, structural reliability analysis, adaptive Monte Carlo, importance sampling, probability of failure.

1. Introduction

The basic engineering principle is to provide an efficient design that is safe. Eurocode 0 provides minimum safety metrics for the built environment. The metrics are normally stated as safety indices, which may also be expressed as probabilities of failure. For most structural engineering applications, the minimum safety index is 3.8, with a corresponding probability of failure, p_f, of $\leq 7 \times 10^{-5}$. With structural resistance typically defined by material properties (e.g., strength and stiffness) and geometry (i.e., at element and structure levels), and load effects determined from a range of environmental factors, it is clear that the amount by which the structural resistance exceeds the correspond load effects (e.g., stress, strain, displacement, etc.) is uncertain, and changes over time (e.g., Fig. 1).

In existing application-specific Eurocodes (e.g., 2 onwards), the margin between resistance and load effect is maintained by the use of partial factors (normally $\geq$ 1.0), either dividing (resistance) or multiplying (load effect) nominal values. The partial factors are derived from a combination of the (target)

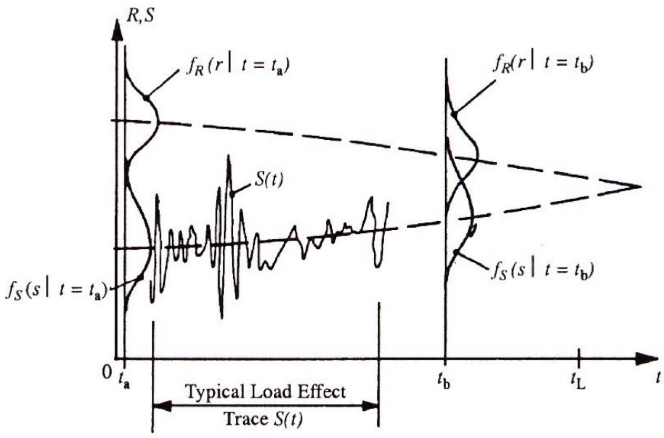

Figure 1: Resistance (R), and load effect (S) varying with time. Acknowledgement [2]

safety index (e.g., 3.8), and the statistics associated with the resistance and load effect design variables. It is the combination of statistically low resistance with statistically high load effect that determines the safety index or the probability of failure. As with all basic design problems, the challenge is selecting the material and geometry combination that provides sufficient resistance, where "sufficient" is defined in Eurocode 0 via the minimum safety index. The analysis challenge is a structural reliability problem - calculating the safety index or probability of failure for a candidate design resistance given a prior load effect. In short, we seek the probability that the resistance is $<$ the load effect, for all possible combinations (e.g., Fig. 2).

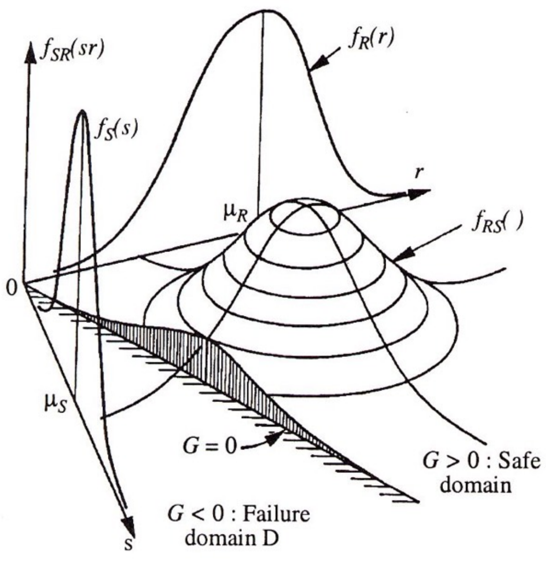

Figure 2: Basic structural reliability problem. Acknowledgement [2]

The ultimate limit state design of cable and membrane structures is normally based on a combination of factored loads and a type of permissible stress principle, with the material strength reduced by factors considering phenomena such as tear propagation and the effects of environmental factors and impacts. A similar approach is adopted for serviceability criteria. The design choices are focussed on material strength, prestress, and surface geometry, with elastic stiffness playing a more minor role.

Encompassing cable and membrane structure design within a Eurocode framework, at first sight, might not necessarily be expected to generate significant complications. After all, application of the framework to steel, concrete, and timber structural systems, for example, in which partial factors are applied to load effect and resistance (e.g., material and section properties), appears straightforward, and account for inherent (aleatoric) uncertainties. Perhaps less obvious, is that the calibration of the partial factors is undertaken within the context of the overarching Eurocode 0, where structural safety is defined in terms of a minimum probability of failure (or safety index) rather than a direct measure of a physical engineering quantity (e.g., stress).

Predicting the probability of failure (or safety index, in standard normal space, N(0,1)) requires the writing of a performance statement (limit state) in terms of resistance and load effect, with their relative proximity a measure of the probability. Unlike geometrically linear structural systems (which may also be materially non-linear, i.e., elasto-plastic) the definition of resistance and load effect for geometrically nonlinear structural systems (e.g., cable and membrane structures) may be neither clear nor constant. For example, prestress contributes to the total stress as a load effect, but at the same time provides stiffness, and is also, therefore, a resistance term. Different partial factors could be applied to each component, but this requires an understanding of how they contribute to the complete structural system, and completion of a calibration process. To ensure rigour, both demand some form of structural reliability analysis.

The (deterministic) structural analysis of a tension structure is computationally expensive. A reliability analysis adds a further level of computational cost and complexity. First-order reliability methods (FORM) may provide faster solution options, but require limit state derivative information (which may not be available), and assume a linear limit state (which is not normally the case for nonlinear structural systems). Second order methods (SORM) improve on the limit state approximation, but demand curvatures. The minimum probability of failure permitted by Eurocode 0, 7×10^{-5}, requires an infeasible 10^7–10^8 structural analyses per limit state to achieve a reasonable level of accuracy. Adaptive Monte Carlo simulation with importance sampling may offer a potentially effective, practical solution approach to the problem of calculating the probability of failure (and safety index) of highly non-linear tension structures.

This paper outlines the principles of adaptive Monte Carlo simulation with importance sampling, demonstrating how small probabilities can be estimated using low numbers of computations. The method is applied to the analytical analysis of a suspended cable, illustrating how probabilities of failure consistent with the expectations of Eurocode 0 may be estimated. The aim is to be able to predict the probability of failure using 10s of structural analyses rather than millions.

2. Exemplar mathematical model of a highly nonlinear tension structure

As a 2-D continuum in 3-D space, a tensioned membrane represents a complex solid mechanics computational analysis problem, defined by multi-dimensional differential equations. Some simplifications are possible for a very small number of special cases, but, in general, the solution of the problem differential equations requires the use of (approximating) numerical methods, such as the finite element method, and alternative variants. The scope of this paper does not include consideration of these methods. A cable provides a useful 1-D (in 2-D space) analogue to some of the behaviour of a tensioned membrane when subjected to external loading. In particular, it describes strongly geometrically nonlinear behaviour, with aspects very similar to those of its 2-D membrane counterpart along the principal axes. Analytical solutions describing displacement and axial tension have been developed for suspended cables subjected to distributed and concentrated loads [3]. The adoption of an analytical solution as an exemplar mathematical model containing the required geometric nonlinearities, permits validation using basic Monte Carlo simulation, which would otherwise be infeasible for a continuum-based membrane model. The aim in this paper is to establish if a tractable (e.g., total time of computation) approach can be identified to calculate probabilities of failure of the order of 10^{-5}. This capability is required if the principles of Eurocode 0 are to be applied to the design of tensioned membrane structures.

2.1. Response of a cable to a point load

The exact analysis of simple suspended cable problems leads to cumbersome solution methods. Simplifications can be made when the unloaded (self-weight only) profile of the cable is flat with a relatively low sag. Importantly for current study, this approximate theory provides explicit, consistent methods for finding the static response to applied loads accurate to the third order of small quantities. We consider the profile of a uniform cable hanging under its own weight between two supports positioned at the same level. If this profile is flat, so that the ratio of sag to span is ≤1:8, the differential equation governing vertical equilibrium of an element is accurately specified by,

$$H\frac{d^2z}{dx^2} = -mg, \tag{1}$$

for which the relevant solution is,

$$\mathbf{z} = \frac{1}{2}\mathbf{x}(1-\mathbf{x}), \tag{2}$$

with the non-dimensionalised variables; $\mathbf{x} = x/l$, in which l is the span of the cable, and $\mathbf{z} = z/(mgl^2/H)$, with mg the cable weight per unit length, and H the horizontal component of cable tension, given by,

$$H = \frac{mgl^2}{8d}, \tag{3}$$

and d is the sag of the cable, and the cable length is,

$$L = \int_0^l \left\{1 + \left(\frac{dz}{dx}\right)^2\right\}^{\frac{1}{2}} dx = l\left\{1 + \frac{8}{3}\left(\frac{d}{l}\right)^2 - \frac{32}{5}\left(\frac{d}{l}\right)^4 ...\right\}. \tag{4}$$

In these calculations, the effects of the cable stretch are included. The bending stiffness of the cable is assumed to be insignificant. As a starting point, we consider a cable with a point load applied at a distance x_1 from the left support (Fig. 3). Assuming that deformations are sufficiently small so that the profile remains shallow, vertical equilibrium at a cross-section of the cable requires that,

$$(H+h)\frac{d}{dx}(z+w) = P\left(1 - \frac{x_1}{l}\right) + \frac{mgl}{2}\left(1 - \frac{2x}{l}\right), \tag{5}$$

for $0 \leq x < x_1$, where w is the additional vertical cable deflection and h is the increment in horizontal component of cable tension generated by the application of P. Expanding and simplifying 5 (self-weight terms cancel), then,

$$(H+h)\frac{dw}{dx} = P\left(1 - \frac{x_1}{l}\right) - h\frac{dz}{dx}, \tag{6}$$

with a similar definition obtained for $x_1 \leq x < l$. Integrating 6 directly whilst satisfying the boundary conditions, the dimensionless equation for the additional vertical deflection is obtained as,

$$\mathbf{w} = \frac{1}{(1+\mathbf{h})}\left\{(1-\mathbf{x}_1)\,\mathbf{x} - \frac{\mathbf{h}}{2\mathbf{P}}\mathbf{x}\,(1-\mathbf{x})\right\}, \qquad 0 \leq \mathbf{x} < \mathbf{x}_1, \tag{7}$$

with, $\mathbf{w} = w/(Pl/H)$, $\mathbf{h} = h/H$, and $\mathbf{P} = P/mgl$. To complete the solution $\mathbf{h}$ is required. Hooke's law is used to link changes in the cable tension to changes in the cable geometry when the cable is displaced from its original equilibrium (self-weight) profile. If ds is the original length of an element of the cable, and EA the axial stiffness, then the cable equation for the element is,

$$\frac{h\,(ds/dx)^3}{EA} = \frac{du}{dx} + \frac{dz}{dx}\frac{dw}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2. \tag{8}$$

Integrating 8 by parts, and making use of 7, yields a dimensionless cubic equation for $\mathbf{h}$, as,

$$\mathbf{h}^3 + \left(2 + \lambda^2/24\right)\mathbf{h}^2 + \left(1 + \lambda^2/12\right)\mathbf{h} - \lambda^2 \mathbf{x}_1\left(1 - \mathbf{x}_1\right)\mathbf{P}\left(1 + \mathbf{P}\right)/2 = 0, \tag{9}$$

with $\lambda^2 = (mgl/H)^2 l/(HL_e/EA)$, and $L_e \approx l(1+8(d/l)^2)$. 9 has only one real root (Descarte's rule), which is the required value of $\mathbf{h}$. $\mathbf{h}$ is multiplied by H to give the increment of horizontal component of axial cable force, which, if evaluated at centre span, and given symmetrical loading and support conditions, corresponds to the tensile force in the cable.

The purpose of providing a relatively detailed background to this exemplar mathematical model, is to highlight the complexity and nonlinearity (in the form of the roots of a cubic equation) of the solution for a cable, the level of which is increased when considering a continuum-based membrane. It should be further noted at this stage that the equilibrium configuration of the cable with a point load is not differentiable at the position of the load.

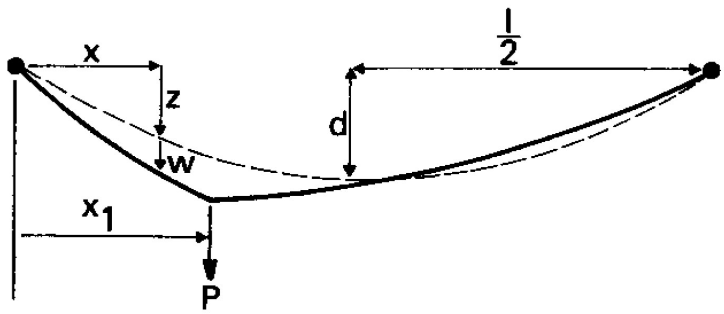

Figure 3: Definition diagram for a point load on a cable. Acknowledgement [3]

2.2. Response of a cable to a uniformly distributed load

Following the principles of the point load case, a similar mathematical model is described for a cable with a uniformly applied load, of intensity p per unit length of span, applied from $x = x_2$ to $x = x_3$ (see Fig. 4). Most relevant to structural design, of the three different regions of the span, the dimensionless equation for the additional vertical cable deflection for the loaded part is provided, and written as,

$$\mathbf{w} = \frac{1}{(1+\mathbf{h})}\left[\frac{1}{2}\left(\mathbf{x}_3^2 - \mathbf{x}_2^2\right)(1-\mathbf{x}) - \frac{\mathbf{h}}{2\mathbf{p}}\mathbf{x}\,(1-\mathbf{x})\right], \qquad \mathbf{x}_2 \leq \mathbf{x} \leq \mathbf{x}_3 \tag{10}$$

with, $\mathbf{w} = w/(pl^2/H)$, $\mathbf{h} = h/H$, $\mathbf{x} = x/l$, and $\mathbf{p} = p/mg$.

For a uniformly distributed load (including part-span), dw/dz is continuous along the span. In this case, the equivalent of 8 becomes,

$$\frac{hL_e}{EA} = -\int_0^l \left(\frac{d^2z}{dx^2} + \frac{1}{2}\frac{d^2w}{dx^2}\right) w\,dx, \tag{11}$$

with the corresponding dimensionless cubic in **h** obtained as,

$$\mathbf{h}^3 + \left(2 + \frac{\lambda^2}{24}\right)\mathbf{h}^2 + \left(1 + \frac{\lambda^2}{12}\right)\mathbf{h} - \frac{\lambda^2}{2}\left\{\frac{1}{2}\left(\mathbf{x}_3^2 - \mathbf{x}_2^2\right) - \frac{1}{3}\left(\mathbf{x}_3^3 - \mathbf{x}_2^3\right)\right\}\mathbf{p} \\ -\frac{\lambda^2}{2}\left\{\frac{1}{3}\left(\mathbf{x}_3^3 - \mathbf{x}_2^3\right) - \mathbf{x}_2^2\left(\mathbf{x}_3 - \mathbf{x}_2\right) - \frac{1}{4}\left(\mathbf{x}_3^2 - \mathbf{x}_2^2\right)^2\right\}\mathbf{p}^2 = 0 \tag{12}$$

Again, h can be calculated using the single positive root obtained from 12. For a UDL acting over the full span, $\mathbf{x}_2 = 0$, with $\mathbf{x}_3 = l$. As expected, 12 converges towards 9 as $\mathbf{x}_3 - \mathbf{x}_2 \to 0$. This concludes the outline of the mathematical model.

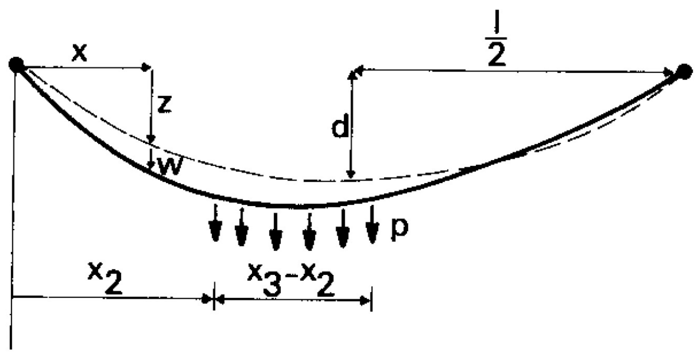

Figure 4: Definition diagram for a uniformly distributed load on a cable. Acknowledgement [3]

3. Solving the structural reliability problem using adaptive Monte Carlo simulation with importance sampling

3.1. Basic Monte Carlo simulation

Monte Carlo simulation (MCS) performs numerical tests on the limit stage, $G(\overline{x})$, with $\overline{x}$ the vector of design variables (e.g., uncertain model parameters such as material properties, loading, etc.), by sampling seeking $\overline{x}$ and evaluating $G(\overline{x})$ to identify $G(\overline{x}) < 0$. The number of instances of $G(\overline{x}) < 0$ is summed and divided by the total number of samples to give the probability that $G(\overline{x}) < 0$, which corresponds to the probability of failure, p_f given the definition of G. Expressed mathematically,

$$p_f = \int_{-\infty}^{G(\overline{x})<0} f(\overline{x})d\overline{x}, \tag{13}$$

where $f(\overline{x})$ is the joint probability density function (PDF) describing the basic variables $\overline{x}$. The integral upper limit, $G(\overline{x}) < 0$, in 13 can be conveniently implemented with the use of the indicator function, $I(\overline{x})$, such that $I(\overline{x}) = 1$ when $G(\overline{x}) < 0$, and zero otherwise, giving,

$$p_f = \int_{-\infty}^{+\infty} I(\overline{x})f(\overline{x})d\overline{x}. \tag{14}$$

Noting that the mean, μ, of a set of numbers, x, with PDF $f(x)$, is defined as,

$$\mu = \int_{-\infty}^{+\infty} xf(x)dx, \tag{15}$$

comparing 14 with 15, then we obtain the formalised definition (assuming all outcomes are normally distributed) for a finite number of samples, n,

$$p_f \approx \mu\left(I(\overline{x})\right) \approx \frac{1}{n}\sum_{i=1}^{n} I(\overline{x}). \tag{16}$$

3.2. A refinement of basic Monte Carlo simulation

The indicator function $I(\overline{x})$ is rather crude in that it doesn't provide any information about the proximity of the solution to the limit state, and, therefore, no corresponding refinement of the estimate of p_f. An improved estimate of p_f may be made by incorporating the limit state constraint (written in terms of resistance R and load effect S) using the joint PDF, f_{RS}, and the cumulative density function (CDF) of R, F_R, as in,

$$p_f = p\left(R - S < 0\right)) = \int_{-\infty}^{+\infty}\int_{-\infty}^{(R-S)<0} f_{RS}(\overline{x})d\overline{x} = \int_{-\infty}^{+\infty} F_R(\overline{x})f_S(\overline{x})d\overline{x}. \tag{17}$$

Comparing 14 and 16, with 17, then,

$$p_f = \int_{-\infty}^{+\infty} F_R(\overline{x})f_S(\overline{x})d\overline{x} \approx \mu\left(F_R(\overline{x})\right) \approx \frac{1}{n}\sum_{i=1}^{n} F_R(\overline{x}), \tag{18}$$

with samples taken from $f_S(\overline{x})$.

3.3. Monte Carlo simulation with importance sampling

In basic Monte Carlo simulation, samples are naturally centred around the means of the variables $\overline{x}$. Using importance sampling, this location is shifted to the 'most probable (failure) point' (MPP), e.g., the location on the limit state describing the most onerous values of resistance and load effects, leading to failure (Fig. 5). The importance sampling function, $h_v(x)$ multiplies and divides the integral in 17,

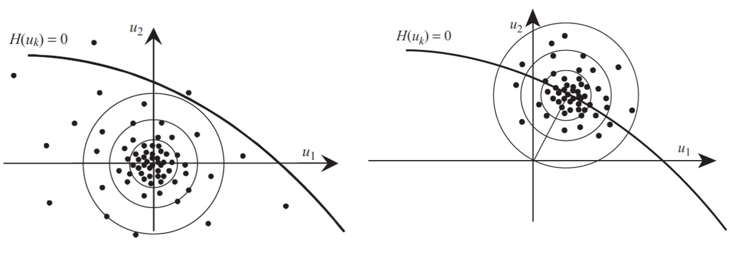

Figure 5: Sampling examples for basic and importance sampling MCS, respectively.

leaving the numerical value of p_f unchanged, but importantly, shifting the sampling of the variables from f_s to h_v, as in,

$$p_f = \int_{-\infty}^{+\infty}\left[F_R\left(\overline{v}\right)\frac{f_S\left(\overline{v}\right)}{h_v(\overline{v})}\right]h_v(\overline{v})d\overline{v} \approx \mu\left(F_R(\overline{v})\frac{f_S(\overline{v})}{h_v(\overline{v})}\right) \approx \frac{1}{n}\sum_{i=1}^{n}\left(F_R(\overline{v})\frac{f_S(\overline{v})}{h_v(\overline{v})}\right) \tag{19}$$

The inclusion of $F_R(\overline{x})$ in 19 provides the potential for greater numerical efficiency (e.g., the need for fewer samples) compared with basic MCS. However, the availability of $F_R(\overline{x})$ is not assured. For the current cable example (and tensioned membranes in general), the coupling of terms in the resistance and load effects (e.g., displacement and axial load/stress) prevent the explicit analytical definition of $F_R(\overline{x})$. A numerical approximation of $F_R(\overline{x})$ could be generated, but this would require a large number of samples, defeating the aim. In this case, referring to 14, 19 will be used in the less refined form,

$$p_f = \int_{-\infty}^{+\infty} \left[I(\overline{v}) \frac{f_S(\overline{v})}{h_v(\overline{v})} \right] h_v(\overline{v}) d\overline{v} \approx \mu \left(I(\overline{v}) \frac{f_S(\overline{v})}{h_v(\overline{v})} \right) \approx \frac{1}{n} \sum_{i=1}^{n} \left(I(\overline{v}) \frac{f_S(\overline{v})}{h_v(\overline{v})} \right) \tag{20}$$

The statistics of $h_v(\overline{x})$ (e.g., mean, $\mu(\overline{x})$, and standard deviation, $\sigma(\overline{x})$, [if assumed to be normally distributed]), need to be chosen. Their choice impacts the accuracy of the predicted probability, p_f, and rate of convergence. Both $\mu(\overline{x})$ and $\sigma(\overline{x})$ may be found to be of similar significance. The design point (an estimate of the MPP) from a first-order reliability analysis (FORM) can provide a reasonable candidate for $\mu(\overline{x})$. Since FORM generates only a single design point, standard deviation information is not available. The standard deviations from the basic variables $\overline{x}$ may be tentatively adopted for $h_v(\overline{x})$. The importance sampling function may be **adapted** using the values of the design values $\overline{x}$ for which $G(\overline{x}) < 0$ to generate new values of $\mu(\overline{x})$ and $\sigma(\overline{x})$ to describe $h_v(\overline{x})$.

FORM typically transforms the limit state into standard normal space (e.g., $N[0, 1]$), and seeks to find the minimum distance from the origin (e.g., the means of the transformed $\overline{x}$) to the limit state. This distance equates to the safety index, β, from which the p_f can be calculated using the inverse CDF (e.g., $p_f = \Phi(-\beta)$). Effectively, FORM is an optimisation problem. Whilst several methods can be used, the most common are gradient-based approaches, requiring derivatives of the limit state function with respect to the design variables $\overline{x}$. If analytical derivatives are not readily available, as will typically be the case for analysis codes used for the design of tensioned membranes, finite difference approximations may provide acceptable alternatives. With the analytical cable solution requiring the roots of a cubic equation, finite differences are used to estimate the gradients of the limit state in this case.

4. Estimating p_f of a suspended cable

A cable of span $l = 100m$ is suspended between supports at the same level. Young's modulus, E, and cross-sectional area, A, are assumed to be 1.5×10^8 kN/m^2 and 5×10^{-4} m^2, respectively, giving $EA = 7.5 \times 10^4$ kN. The mass of the cable, m, is 4 kg/m. The calculation point for h and w, x, is taken as mid-span. In what follows, two input parameters are assumed to be basic variables, and the resistance of the cable is taken to be deterministic. This simple scenario permits a clearer initial assessment of the approach to calculate p_f at this stage.

4.1. Uniformly distributed load

A uniformly distributed load (UDL) is applied to the full span of the cable, such that $\mathbf{x}_2 = 0$ and $\mathbf{x}_3 = 100m$. Load p is assumed to be a basic variable, with a mean value of 0.5 KN/m and a coefficient of variation (CoV) of 10%, such that $p =: N[0.5, 0.05]$ kN/m. The initial sag of the cable, d, is also assumed to be uncertain, with $\mu(p) = 2$ m and a CoV of 5%, giving $d =: N[2, 0.1]$ m. The limit state is based on the mid-span cable force, with a deterministic resistance of 239 kN.

The starting point is to estimate the MPP from FORM, with derivatives of the limit state estimated using forward finite differences and 1% perturbations. Initial values for the standard normal variables ($y_1 \to p$, $y_2 \to d$) were assumed to be at the means. The large magnitude sensitivities of the limit state with respect to the basic variables resulted in slow convergence, which was accelerated by extrapolating estimates of the standard normal variables. With (discontinuously) converged values for y_1 and y_2 of 3.80 and -0.43, respectively, after 11 iterations, the corresponding limit state, $g(\overline{y})$, was close to the required value of zero (0.06 kN) compared with 47.4 kN at the mean values of the basic variables (e.g., $\overline{y} = \overline{0}$). Normalised derivatives indicated that uncertainty in the applied load dominated the failure of the cable, in this case. The associated design point, (p^*, d^*) was $(0.69\text{ kN/m}, 1.957\text{ m})$ with a safety index of $\beta = 3.83$, giving $p_f \approx 6.4 \times 10^{-5}$.

For the Monte Carlo simulation with importance sampling (MCS-IS), the mean values in $h_v(\overline{v})$ are assumed to be (p^*, d^*) from the FORM analysis (e.g., $(0.69\text{ kN/m}, 1.957\text{ m})$), and in the absence of other data, the CoVs of the basic variables are retained to calculate the corresponding standard deviations.

At this point, it is worth noting that for basic Monte Carlo simulation (bMCS), the minimum probability that can be calculated is $1/n$, with n the number of samples (e.g., for $n = 100$, $p_{min} = 0.01$, etc.). Therefore, for, e.g., the EC0 target probability of 7×10^{-5}, the minimum number of samples to identify a single instance of $G < 0$ is $1/7 \times 10^{-5} = 15,000$, with the accurate estimate of p_f requiring several orders of magnitudes more (as evidenced below).

To explore the stated aim of this study, just **20** samples are assumed with MCS-IS. (20 analyses remains a large number in the field of tensioned membranes, but is a significant reduction over bMCS). A set of 20 samples of the basic variables based on the $h_v(\overline{v})$ described above, generated the estimate $\mathbf{p_f \approx 3.64 \times 10^{-5}}$.

This result appears to be infeasible/improbable given observations in the preceding paragraph. However, comparing 16 with 20, it is clear that the ratio $\frac{f_S(\overline{v})}{h_v(\overline{v})}$ plays a significant role in scaling the outcomes from a Monte Carlo-type simulation. As a ratio of probabilities, $\frac{f_S(\overline{v})}{h_v(\overline{v})}$ may be viewed as a type of likelihood. $\overline{v}$ is the vector of samples of the basic variables (p and d in the current example), obtained from a uniform normal distribution using (p^*, d^*). $h_v(\overline{v})$ is the probability of obtaining the sampled values of $\overline{v}$. These probabilities are typically in the range $10^{-1} - 10^{-2}$ as expected given the mean and standard deviation of $\overline{v}$ are used to define $h_v(\overline{v})$. In contrast, $f_S(\overline{v})$ is the probability of $\overline{v}$ given the statistics for S. Since $\overline{v}$ is sampled around the MPP, $f_S(\overline{v})$ has a much broader range of smaller probabilities; e.g., $10^{-5} - 10^{-10}$. $f_S(\overline{v})$, provides the opportunity to return much smaller values of probability despite very small numbers of samples. For the first set of 20 samples, 11 generated instances of $G < 0$. Two further sets of 20 samples returned 10 and 8 instances, with corresponding failure probabilities of $p_f \approx 1.71 \times 10^{-5}$ and $p_f \approx 5.98 \times 10^{-5}$, respectively, giving an average over the three sets of $p_f \approx 3.78 \times 10^{-5}$. These outcomes are based on mean values for $h_v(\overline{v})$ taken from (p^*, d^*) with standard deviations calculated using the original data CoVs. The mean and standard deviation for $h_v(\overline{v})$ can be re-estimated from the values of (p, d) that generated instances of $G < 0$ from the FORM starting point. This adaptive MCS-IS aims to improve the estimated p_f by refining the importance sampling around the predicted MPP. A further six simulations of 20 samples each combine to give an updated $p_f \approx 3.87 \times 10^{-5}$. The associated CoV of 124.9% indicates a high level of variability associated with such a small sample size.

A basic MCS (programmed in MATLAB) with $2,000,000 \leq n \leq 200,000,000$ (Table 1) suggests that the truer $p_f \approx 7.00 \times 10^{-5}$ with CoV $\approx 5.0\%$. Comparing predictions from FORM ($p_f \approx 6.4 \times 10^{-5}$), adaptive MCS-IS ($p_f \approx 3.87 \times 10^{-5}$), and basic MCS ($p_f \approx 7.00 \times 10^{-5}$), it would be sufficient to adopt the FORM solution, in this case. The difference between the FORM and basic MCS predictions indicate a small amount of nonlinearity in the limit state, with the MCS-IS scheme with only 20 samples unable to resolve this detail.

4.2. Patch load

The UDL is replaced by a short (1m long) patch load (with $\mathbf{x}_2 = 49.5$ and $\mathbf{x}_3 = 50.5m$) of uncertain mean intensity 20 kN/m and a CoV $= 10\%$. The resistance of the cable, R, is assumed to be deterministic, with $R = 182.9$ kN. All remaining parameters are the same as the UDL case. The response of the

Table 1: Monte Carlo simulation convergence - UDL

n	p_f	CoV
$20 \times 100,000 = 2,000,000$	6.5000×10^{-5}	34%
$100 \times 20,000 = 2,000,000$	7.1000×10^{-5}	95%
$20 \times 1,000,000 = 20,000,000$	7.2350×10^{-5}	10%
$20 \times 10,000,000 = 200,000,000$	7.0105×10^{-5}	4.8%
$20 \times 10,000,000 = 200,000,000$ (repetition)	6.9990×10^{-5}	5.2%

cable to the patch load is a combination of a point load and UDL as defined by 12. In this case, using FORM fails to identify the MPP. Starting at the mean values of the basic variables, $G = 37.4kN$, which is less than the UDL case, suggesting a potentially higher p_f. In reverse to the UDL case, the sensitivities of G with respect to the basic variables are dominated by the initial cable sag, d, and not p. At $G = 0.91$ kN, and, therefore, close to limit state, $\beta = 8.8$, equivalent to an equally unrealistic $p_f \approx 8 \times 10^{-19}$. The failure of the FORM algorithm is in contrast to the UDL case, with a suggestion that the solution for the increment of the horizontal component of the cable force, h, is not sufficiently smooth, and, therefore, non-differentiable. Selecting a realistic value of β from the FORM iterations ($\beta = 3.7$), the corresponding (assumed approximation to the) MPP gives $(p, d)^* = (21.4\,\text{kN}, 1.63\,\text{m})$ on which to base $h_v(\overline{v})$ with the original CoVs for the MCS-IS. From 20 samples, this $h_v(\overline{v})$ generated only a single instance of $G < 0$. Further exploration of $h_v(\overline{v})$ provided a highest estimate of $p_f \approx 1.54 \times 10^{-6}$, a considerable underestimate of the truer $p_f \approx 6.96 \times 10^{-5}$ (MCS with $20 \times 10,000,000$, CoV 4.4%).

Noting that confidence in the FORM estimate of the MPP for this case is low (not reliable or correct), sampling $h_v(\overline{v})$ based on a uniform distribution in $N(0,1)$ with an uncertain (highly) MPP, may compound an inability to predict the p_f with sufficient accuracy when using very few samples. In this context, Latin Hypercube Sampling (LHS), which improves the coverage of the input space, using a stratified scheme that is rather more disruptive in generating samples compared with the uniform $N(0,1)$ approach [4], may prove an effective alternative scheme. Sampling $h_v(\overline{v})$ using adaptive LHS (initial ($\mu_p = 26, \mu_d = 1.45$, CoVs $= 5\%$) with a total of 26 samples (of which 15 indicated $G < 0$), the average $p_f \approx 8.94 \times 10^{-5}$. Given the low number of samples, this represents a very reasonable prediction of the truer value of $p_f \approx 6.96 \times 10^{-5}$ obtained from $200,000,000$ basic Monte Carlo simulations. It should be noted that the use of arbitrary initial estimates for $h_v(\overline{v})$ may lead to invalid $f_S(\overline{v})/h_v(\overline{v}) > 1$.

5. Conclusions

This study demonstrates the potential of Monte Carlo simulation with importance sampling to predict the p_f of highly geometrically nonlinear structures using very small numbers of limit state (design scenario) evaluations. Care must be taken in defining $h_v(\overline{v})$ (with the MPP from FORM not necessarily providing a good candidate starting point) and in the application of the method used to sample $h_v(\overline{v})$, particularly when defining μs and CoVs.

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