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On the decomposition of degenerate subspaces of symmetric structural configurations

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Abstract

Structural configurations exhibiting symmetry properties (and hence belonging to a symmetry group) may be conveniently studied using the mathematics of group theory, which allows the space of the problem to be decomposed into independent symmetry-adapted subspaces. Within the domain of structural mechanics, group theory has been successfully employed to simplify problems of the bifurcation, stability, statics, kinematics and vibration of symmetric configurations of space frames, space truss domes, double-layer and triple-layer space grids, plates and cable-net systems. Besides significantly reducing computational effort, group theory affords deeper insights on structural behaviour, and a better understanding of complex structural phenomena (for instance, it explains why certain natural frequencies repeat in symmetric vibrating systems). The key to group-theoretic simplification is the decomposition of the space of the symmetric problem into independent subspaces that are spanned by symmetry-adapted variables, allowing the problem to be broken down into smaller independent problems that are easier to study, or easier to analyse. Symmetry-adapted variables are generated by applying special operators (called *idempotents* of the symmetry group) on the normal variables of the problem. However, for *degenerate* subspaces of a symmetry group (i.e. subspaces associated with doubly-repeating or multiply-repeating solutions), the associated idempotents do not sufficiently decompose the problem. The purpose of this paper is to propose, for certain symmetry groups, new operators that fully decompose such subspaces. These operators have never been reported in the literature before. We believe this contribution significantly advances group-theoretic computational analysis.

Keywords: computational analysis, symmetry, group theory, vector-space decomposition, idempotent, symmetry-adapted variable, eigenvalue analysis

1. Introduction

Symmetry is very common in structural engineering and architecture. Besides its aesthetic appeal, symmetry can enhance the functionality of space. From a structural point of view, symmetry can be taken advantage of to simplify the analysis of the system, or to reduce the costs of assembly of the system. However, symmetry also attracts complications in structural behaviour, such as the occurrence of multiple critical points in bifurcation analysis (where two or more eigenvalues vanish simultaneously), and the coincidence or near-coincidence of eigenvalues in problems of the vibration or buckling of structures, both of which pose difficulties of numerical ill-conditioning of solution procedures in computational schemes [1, 2]. Suitable tools are needed to facilitate the study of structural configurations with higher-order symmetries, and better understand the associated complex phenomena.

The set of symmetry elements describing the symmetry of a physical configuration constitutes a symmetry group. Group theory provides the mathematical tools for the study of such systems [3, 4]. This allows the space of the problem to be decomposed into independent symmetry-adapted subspaces. Within the domain of structural mechanics, group theory has been successfully employed to simplify the study of the bifurcation of space trusses [1, 5], the statics of space frames [6, 7] and pin-jointed trusses [8], the vibration of cable nets [9, 10], layered space grids [11] and plates [12], the kinematics of skeletal structures [13-15], and the stability of frames [16, 17] and origami [18]. Applications in computational structural mechanics were highlighted in a survey that was conducted fifteen years ago [19].

Besides significantly reducing computational effort, group theory affords deeper insights on structural behaviour, and a better understanding of complex structural phenomena (for instance, it explains why certain natural frequencies repeat in symmetric vibrating systems [20, 21]). The key to group-theoretic simplification is the decomposition of the space of the symmetric problem into independent subspaces that are spanned by symmetry-adapted variables, allowing the problem to be broken down into smaller independent problems that are easier to study, or easier to analyse. By separating the computation of coincident eigenvalues into independent subspaces that "do not see each other", group theory also circumvents the numerical problems associated with computing closely-spaced or coincident solutions in the full space of the problem [1, 2]. Group theory effectively *untangles* the symmetries.

According to the representation theory of symmetry groups [3, 4], each independent symmetry-adapted subspace S is associated with an irreducible representation Γ of the symmetry group; if the symmetry group has k irreducible representations, then the number of independent symmetry-adapted subspaces will be k. In turn, each irreducible representation $\Gamma^{(i)}$ $(i = 1, 2, \dots, k)$ of the symmetry group is associated with a unique idempotent $P^{(i)}$ $(i = 1, 2, \dots, k)$, which is a very specific linear combination of the symmetry elements of the group, having the special property of nullifying all vectors that do not belong to the subspace $S^{(i)}$ of the irreducible representation $\Gamma^{(i)}$, and selecting only vectors that belong to the subspace $S^{(i)}$. Idempotents of any symmetry group satisfy the relation $P^{(i)}P^{(i)} = P^{(i)}$ for all i. More importantly, they have the property $P^{(i)}P^{(j)} = 0$ if $i \neq j$ (i.e. idempotents of different subspaces are *orthogonal* to each other).

Each subspace has its own characteristic symmetry properties which distinguish it from other subspaces. As an example, the symmetry group C_{1v} describing the symmetry of configurations with one reflection plane (such as a simply supported beam with two equal point loads P equidistant from the centre of the beam) has two irreducible representations $\Gamma^{(1)}$ and $\Gamma^{(2)}$ with corresponding idempotents $P^{(1)} = 0.5(e + \sigma_v)$ and $P^{(1)} = 0.5(e - \sigma_v)$, the symmetry elements $\{e, \sigma_v\}$ denoting the identity operation and reflection operation respectively. The idempotent $P^{(1)}$ and $P^{(2)}$, by operating on the normal variables of the problem, split the space of the problem into a *symmetric* subspace $S^{(1)}$ and an *antisymmetric* subspace $S^{(2)}$ respectively.

Taking the idempotent $P^{(i)}$ corresponding to the irreducible representation $\Gamma^{(i)}$ (and associated with the subspace $S^{(i)}$), and applying this to each of the n normal variables of the problem, we obtain n symmetry-adapted variables, of which say r_i are independent. The r_i independent symmetry-adapted variables may be taken as the basis vectors of subspace $S^{(i)}$. Thus, subspace $S^{(i)}$ is of dimension r_i, where $r_i \ll n$; the sum of the dimensions of all k subspaces is equal to n: that is, $r_1 + r_2 + \cdots + r_k = n$ [9, 11, 20, 22].

For any 1-dimensional irreducible representation $\Gamma^{(i)}$ of a symmetry group (the dimension of $\Gamma^{(i)}$ is given by the first *character* of $\Gamma^{(i)}$ in the character table [3, 4] of the group), the dimension r_i of the associated subspace $S^{(i)}$ is the smallest possible (i.e. no further decomposition of subspace $S^{(i)}$ is possible). However, for an m-dimensional irreducible representation (where m can be 2, 3, 4 or 5), the decomposition yielded by the application of idempotent $P^{(i)}$ results in a subspace $S^{(i)}$ that can still be decomposed further. Such *degenerate* subspaces are associated with repeating solutions (which, in the case of eigenvalue vibration problems, are repeating natural frequencies); the degree of repetition is equal to m. Irreducible representations of dimension 1 or 2 are typically associated with structural configurations belonging to cyclic (C) and dihedral (D) symmetry groups, whereas those of dimension greater than 2 are only encountered in the analysis of tetrahedral (T), octahedral (O) and icosahedral (I) configurations.

Figure 1 shows double-layer grids (in plan and elevation) belonging to symmetry groups C_{3v} and C_{6v}, which characterise configurations with the symmetries of an equilateral triangle (3 rotations and 3 reflections) and a regular hexagon (6 rotations and 6 reflections). One of the three irreducible representations of symmetry group C_{3v} is 2-dimensional (the other two being 1-dimensional), while two of the six irreducible representations of symmetry group C_{6v} are 2-dimensional (the other four being 1-

dimensional) [11]. Thus, symmetry group C_{3v} has two normal subspaces (denoted by $S^{(1)}$ and $S^{(2)}$) and one degenerate subspace (denote by $S^{(3)}$), while symmetry group C_{6v} has four normal subspaces (denoted by $S^{(1)}$, $S^{(2)}$, $S^{(3)}$ and $S^{(4)}$) and two degenerate subspaces (denoted by $S^{(5)}$ and $S^{(6)}$); the degenerate subspaces of both symmetry groups feature doubly-repeating solutions.

Clearly, if further decomposition of a degenerate subspace can be achieved, this would allow the doubly-repeating (or multiply-repeating) solutions of the subspace to be computed more easily. Operators that further decompose the degenerate subspace $S^{(5)}$ of symmetry group C_{4v} (which describe the symmetry of a square) were first presented by the first author in earlier work [11], and have subsequently been successfully applied to the analysis of plates [12], plane grids [22] and plane frames [17], allowing the doubly-repeating eigenvalues of subspace $S^{(5)}$ to be obtained by consideration of only one of its semi-subspaces $S^{(5,1)}$ and $S^{(5,2)}$. As far as the authors are aware of, operators that further decompose the degenerate subspaces of symmetry groups C_{3v} and C_{6v} are not available in the literature. In this paper, we will present unique operators for the automatic decomposition of the degenerate subspace of symmetry groups C_{3v}, and illustrate their application by reference to the double-layer space grid in Figure 1(a). Vibration modes of the space grids shown in Figure 1 were explored in earlier work [11], but without the benefit of these operators. This is the first time such operators have been proposed. Their usefulness go beyond structural mechanics, as they can also be used to simplify problems with this type of symmetry in material science, physics and chemistry. Owing to space constraints, only key results will be presented in this paper; derivational details will be included in a journal version of this paper.

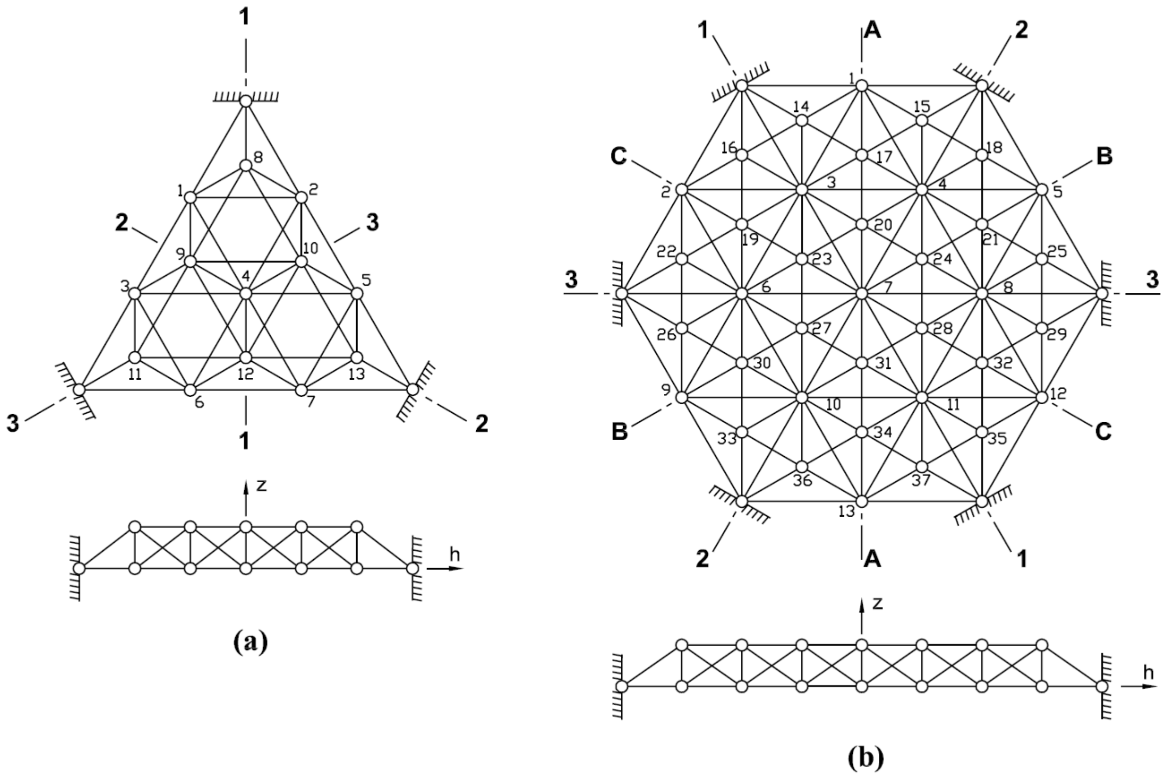

Figure 1: Space grids with C_{nv} symmetry: (a) triangular grid (C_{3v} symmetry); (b) hexagonal grid (C_{6v} symmetry) [11].

2. Idempotents of symmetry group C_{3v}

Symmetry operations are transformations which bring an object into coincidence with itself, and leaves it indistinguishable from its original configuration. In the double-layer grid shown in Fig. 1, nodes 1 to 7 are in the bottom layer, while nodes 8 to 13 are in the top layer, vertically above the centroids of the bottom triangles. The centre of symmetry is at node 4, through which the vertical axis of rotational symmetry of the configuration passes. By reference to the upper diagram of Fig. 1(a), the symmetry operations of group C_{3v}, describing the symmetry of a regular 3-sided polygon, are $\{e, C_3, C_3^{-1}, \sigma_1, \sigma_2, \sigma_3\}$, where e is the identity element (equivalent to a rotation of 2π about the axis of rotational symmetry), C_3 and C_3^{-1} are clockwise and anticlockwise rotations of $2\pi/3$ about the axis of rotational symmetry, while σ_1, σ_2 and σ_3 are reflections in vertical planes $1-1$, $2-2$ and $3-3$ as shown. Idempotents of symmetry group C_{3v} may easily be written down from the character table of the symmetry group [3, 4]. In terms of the symmetry elements of the group, these are as follows:

$$P^{(1)} = \frac{1}{6}(e + C_3 + C_3^{-1} + \sigma_1 + \sigma_2 + \sigma_3) \quad (1)$$

$$P^{(2)} = \frac{1}{6}(e + C_3 + C_3^{-1} - \sigma_1 - \sigma_2 - \sigma_3) \quad (2)$$

$$P^{(3)} = \frac{1}{3}(2e - C_3 - C_3^{-1}) \quad (3)$$

By multiplying out the above operators, it may easily be seen that $P^{(i)}P^{(i)} = P^{(i)}$ for all i $(i = 1, 2, 3)$. Furthermore, the orthogonality property also holds, i.e. $P^{(i)}P^{(j)} = 0$ if $i \neq j$.

3. Basis vectors of the triangular space grid

Let us consider the vertical displacements $\{v_1, v_2, \dots, v_{13}\}$ of concentrated masses at nodes $\{1, 2, \dots, 13\}$ respectively, representing the small transverse motions of the grid as it undergoes free vibration. The vibrating system therefore has $n = 13$ degrees of freedom $\{v_1, v_2, \dots, v_{13}\}$. A conventional lumped-parameter vibration analysis of this system would lead to a 13×13 determinant, the vanishing condition of which results in a 13th-degree characteristic polynomial equation. Solution of the characteristic equation yields 13 eigenvalues (hence natural frequencies of the system), allowing the 13 modes of vibration to be determined. Although this is a relatively small problem, a considerable amount of effort is still required to evaluate the dynamic characteristics of the system (frequencies and modes of vibration). On the other hand, group theory decomposes the 13×13 system matrix into a number of $r \times r$ independent matrices ($r \ll n$), which can be separately solved for all eigenvalues. This separation is achieved by applying idempotents $P^{(1)}$, $P^{(2)}$ and $P^{(3)}$, in turn, upon each of the 13 degrees of freedoms of the system, thus creating three independent subspaces $S^{(1)}$, $S^{(2)}$ and $S^{(3)}$ of the problem.

When the first idempotent $P^{(1)}$ is applied (as an operator) upon $\{v_1, v_2, \dots, v_{13}\}$, we obtain 13 *symmetry-adapted freedoms*, but not all of them are independent. We may select a set of r_1 independent symmetry-adapted freedoms ($r_1 \ll 13$) as the *basis vectors* $\Phi_i^{(1)}$ $(i = 1, 2, \dots, r_1)$ of subspace $S^{(1)}$. Repeating the process using idempotents $P^{(2)}$ and $P^{(3)}$ generates the r_2 basis vectors of subspaces $S^{(2)}$ and the r_3 basis vectors of subspace $S^{(3)}$, respectively. The results for all three subspaces are as follows [11]:

Subspace $S^{(1)}$

$$\Phi_1^{(1)} = v_1 + v_2 + v_3 + v_5 + v_6 + v_7 \quad (4)$$

$$\Phi_2^{(1)} = v_4 \quad (5)$$

$$\Phi_3^{(1)} = v_8 + v_{11} + v_{13} \quad (6)$$

$$\Phi_4^{(1)} = v_9 + v_{10} + v_{12} \quad (7)$$

Subspace $S^{(2)}$

$$\Phi_1^{(2)} = v_1 - v_2 - v_3 + v_5 + v_6 - v_7 \quad (8)$$

Subspace $S^{(3)}$

$$\Phi_1^{(3)} = 2v_1 - v_5 - v_6 \quad (9)$$

$$\Phi_2^{(3)} = 2v_5 - v_1 - v_6 \quad (10)$$

$$\Phi_3^{(3)} = 2v_2 - v_3 - v_7 \quad (11)$$

$$\Phi_4^{(3)} = 2v_3 - v_2 - v_7 \quad (12)$$

$$\Phi_5^{(3)} = 2v_8 - v_{11} - v_{13} \tag{13}$$

$$\Phi_6^{(3)} = 2v_{11} - v_8 - v_{13} \tag{14}$$

$$\Phi_7^{(3)} = 2v_9 - v_{10} - v_{12} \tag{15}$$

$$\Phi_8^{(3)} = 2v_{10} - v_9 - v_{12} \tag{16}$$

Clearly, subspaces $S^{(1)}$, $S^{(2)}$ and $S^{(3)}$ are 4-dimensional ($r_1 = 4$), 1-dimensional ($r_2 = 1$) and 8-dimensional ($r_3 = 8$) respectively. Thus, these subspaces will have 4, 1 and 8 modes of vibration respectively. If the basis vectors of each subspace are plotted as shown in Figure 2, the symmetries of the subspaces become more evident. The plotted values are the coefficients of the v terms in Equations (4–16). Black dots denote positive coefficients of v (downward displacement of the node), while red dots denote negative coefficients of v (upward displacement of the node), the diameter of the dot being proportional to the coefficient.

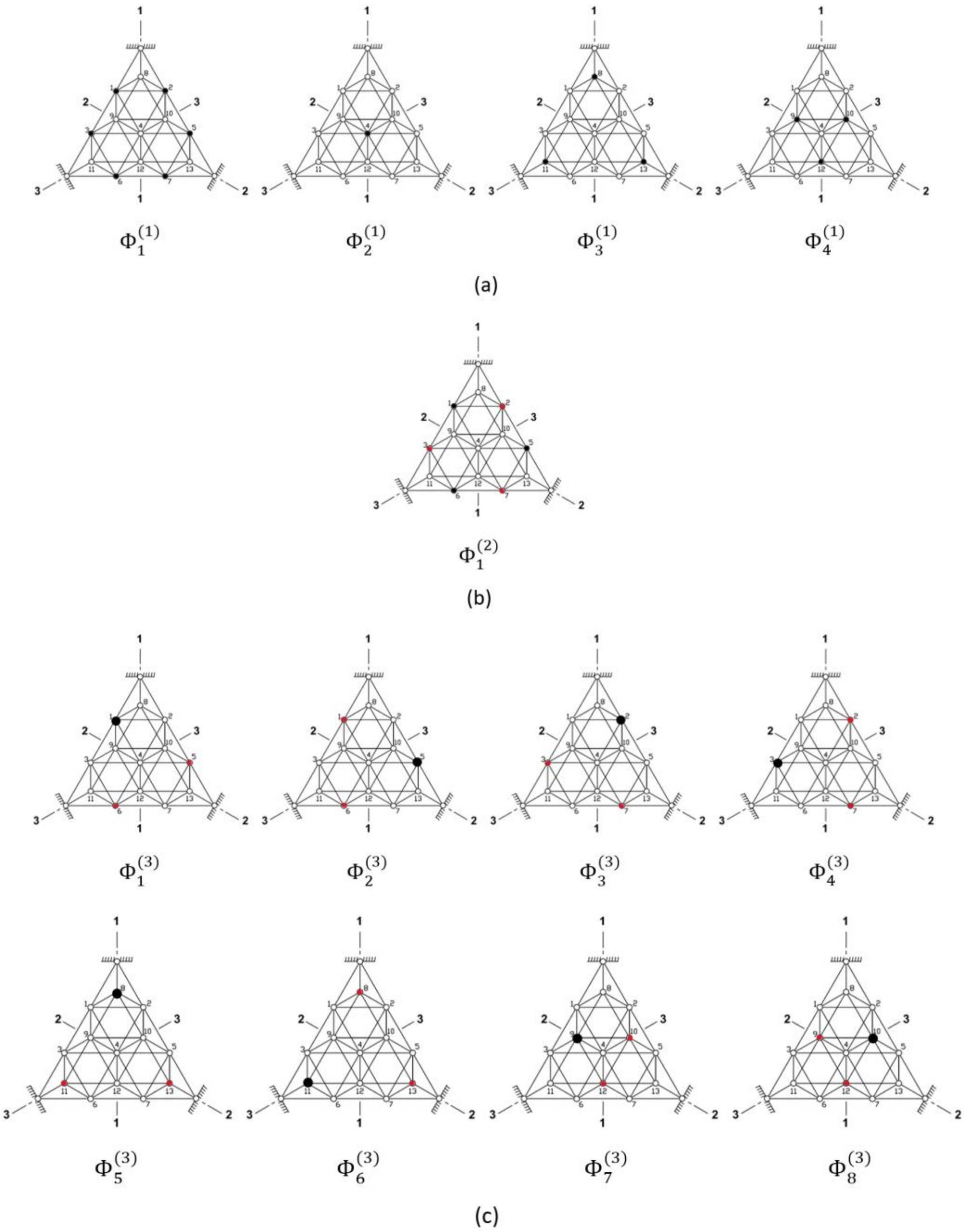

Figure 2: Basis-vector plots of the subspaces of the triangular space grid: (a) subspace $S^{(1)}$; (b) subspace $S^{(2)}$; (c) subspace $S^{(3)}$ [11].

As pointed out earlier, the irreducible representation associated with subspace $S^{(3)}$ is a 2-dimensional representation of the symmetry group C_{3v} [3, 4, 11, 20]. This implies that the eigenvalues in this subspace will be doubly repeating. Thus, the subspace will have only four distinct eigenvalues. However, unless a way can be found of decomposing subspace $S^{(3)}$ further, an 8-dimensional eigenvalue problem (leading to an 8th-degree characteristic equation) will still need to be solved in order to arrive at the four doubly repeating solutions. This requires considerable computational effort. In the next section, a special pair of operators is proposed for the further decomposition of subspace $S^{(3)}$, to reduce computational effort.

4. Special operators for subspace S$^{(3)}$ of symmetry group C$_{3v}$

For problems involving symmetry group C_{3v}, we seek two operators $P^{(3,1)}$ and $P^{(3,2)}$ that are able to subdivide the degenerate subspace $S^{(3)}$ into two smaller subspaces $S^{(3,1)}$ and $S^{(3,2)}$ spanned by linear combinations of the basis vectors of subspace $S^{(3)}$, such that the basis vectors of subspace $S^{(3,1)}$ are orthogonal to those of subspace $S^{(3,2)}$. This would then allow subspaces $S^{(3,1)}$ and $S^{(3,2)}$ to be treated separately. We require these operators to satisfy the following conditions:

$$P^{(3,1)} + P^{(3,2)} = P^{(3)} \tag{17}$$

$$P^{(3,1)}P^{(3,1)} = P^{(3,1)} \tag{18}$$

$$P^{(3,2)}P^{(3,2)} = P^{(3,2)} \tag{19}$$

$$P^{(3,1)}P^{(3,2)} = 0 \tag{20}$$

The first condition is the requirement that the sum of the two special operators must equal the idempotent $P^{(3)}$ as given by Equation (3). The second and third conditions require the two special operators to have the property $P^{(i)}P^{(i)} = P^{(i)}$ common to all idempotents. The last condition requires the two special operators to be orthogonal to each other, to ensure the orthogonality of the basis-vector sets of subspaces $S^{(3,1)}$ and $S^{(3,2)}$.

To preserve the rotational symmetries of the parent idempotent $P^{(3)}$ (see Equation (3)), let each of the sought operators $P^{(3,1)}$ and $P^{(3,2)}$ comprise half of $P^{(3)}$ and a linear combination of reflection elements $\{\sigma_1, \sigma_2, \sigma_3\}$ that is of equal magnitude but of opposite sign (i.e. the linear combination of $\{\sigma_1, \sigma_2, \sigma_3\}$ in $P^{(3,1)}$ must be the negative of that in $P^{(3,2)}$ so that the sum of the two linear combinations is zero). The following expressions for $P^{(3,1)}$ and $P^{(3,2)}$ fulfill all the above conditions:

$$P^{(3,1)} = \frac{1}{6}(2e - C_3 - C_3^{-1} - \sigma_1 - \sigma_2 + 2\sigma_3) \tag{21}$$

$$P^{(3,2)} = \frac{1}{6}(2e - C_3 - C_3^{-1} + \sigma_1 + \sigma_2 - 2\sigma_3) \tag{22}$$

Equations (21) and (22), which have never been proposed in the literature, are the sought special operators for the further decomposition of subspace $S^{(3)}$. They have all the properties of idempotents, so they may be referred to as the idempotents of the semi-subspaces $S^{(3,1)}$ and $S^{(3,2)}$. Let us assume the parent subspace $S^{(3)}$ is of dimension r_3 (this is always an even integer) before it is decomposed. When applied upon the normal variables of a problem, operators $P^{(3,1)}$ and $P^{(3,2)}$ automatically generate the $r_3/2$ basis vectors of subspace $S^{(3,1)}$ and the $r_3/2$ basis vectors of subspace $S^{(3,2)}$ respectively, thus decomposing subspace $S^{(3)}$ into two subspaces that are each of half the size of subspace $S^{(3)}$. In the next section, the two operators will be applied to the further decomposition of subspace $S^{(3)}$ of the triangular space grid.

5. Application of operators to the triangular space grid

Applying the operator $P^{(3,1)}$ (Equation (21)) on each of the freedoms $\{v_1, v_2, \ldots, v_{13}\}$ results in 13 *symmetry-adapted freedoms*, only four of which are independent. Applying the operator $P^{(3,2)}$ (Equation (22)) on each of the freedoms $\{v_1, v_2, \ldots, v_{13}\}$ also results in 13 *symmetry-adapted freedoms*, only four of which are independent. Taking the four independent symmetry-adapted freedoms for each subspace, we obtain the following sets of basis vectors for subspaces $S^{(3,1)}$ and $S^{(3,2)}$ of the triangular grid:

Subspace $S^{(3,1)}$

$$\Phi_1^{(3,1)} = 2v_1 - v_5 - v_6 - v_2 - v_3 + 2v_7 \tag{23}$$

$$\Phi_2^{(3,1)} = 2v_2 - v_7 - v_3 - v_1 - v_6 + 2v_5 \tag{24}$$

$$\Phi_3^{(3,1)} = v_8 + v_{13} - 2v_{11} \tag{25}$$

$$\Phi_4^{(3,1)} = v_9 + v_{12} - 2v_{10} \tag{26}$$

Subspace $S^{(3,2)}$

$$\Phi_1^{(3,2)} = 2v_1 - v_5 - v_6 + v_2 + v_3 - 2v_7 \tag{27}$$

$$\Phi_2^{(3,2)} = 2v_2 - v_7 - v_3 + v_1 + v_6 - 2v_5 \tag{28}$$

$$\Phi_3^{(3,2)} = v_8 - v_{13} \tag{29}$$

$$\Phi_4^{(3,2)} = v_9 - v_{12} \tag{30}$$

The basis vectors of subspaces $S^{(3,1)}$ and $S^{(3,2)}$ are plotted in Figure 3. These subspaces have distinct symmetry properties, as is clearly evident from the plots: all the basis-vector plots of subspaces $S^{(3,1)}$ are *symmetric* about axis $3-3$, while all the basis-vector plots of subspace $S^{(3,2)}$ are *antisymmetric* about axis $3-3$. We therefore expect the vibration modes of subspaces $S^{(3,1)}$ and $S^{(3,2)}$ to exhibit a similar pattern of symmetry properties. Thus, apart from simplifying the computation of actual frequencies and mode shapes of the parent subspace $S^{(3)}$, the operators $P^{(3,1)}$ and $P^{(3,2)}$ also *untangle* the symmetries of the subspace, separating them into C_{1v}-symmetric modes (i.e. modes with one axis of symmetry in plan) and C_1-symmetric modes (i.e. modes with one axis of antisymmetry in plan).

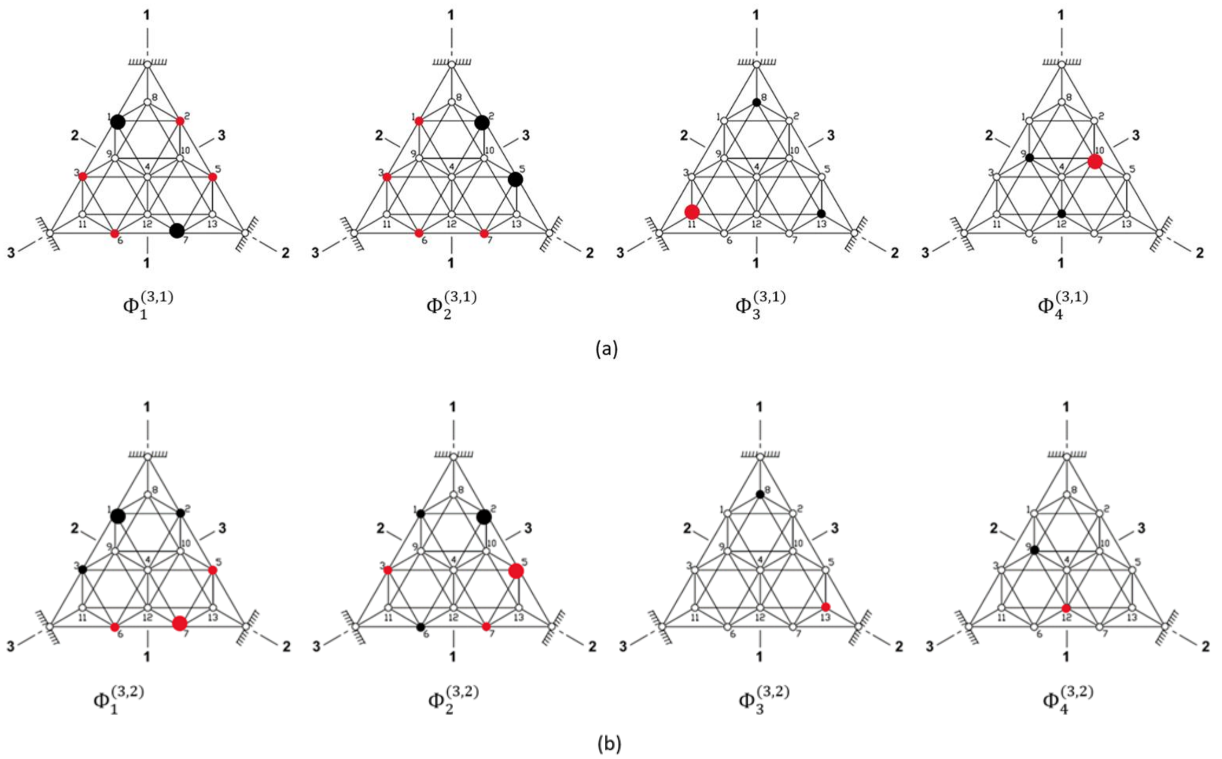

Figure 3: Further decomposition of subspace $S^{(3)}$ of the triangular space grid: (a) Basis-vector plots of subspace $S^{(3,1)}$; (b) Basis-vector plots of subspace $S^{(3,2)}$.

For each mode in subspaces $S^{(3,1)}$, there will be a corresponding mode in subspace $S^{(3,2)}$ that has an identical natural frequency (explaining the phenomenon of doubly-repeating frequencies associated with the parent subspace $S^{(3)}$). However, the basis-vector sets of subspaces $S^{(3,1)}$ and $S^{(3,2)}$ are orthogonal to each other, i.e. $\Phi_i^{(3,1)}\Phi_j^{(3,2)}$=0 for any $i = \{1,2,3,4\}$ and any $j = \{1,2,3,4\}$. For example, writing the coefficients of basis vectors $\Phi_1^{(3,1)}$, $\Phi_1^{(3,2)}$ and $\Phi_2^{(3,2)}$ (see Equations (23), (27) and (28)) as $B_1^{(3,1)}$, $B_1^{(3,2)}$ and $B_2^{(3,2)}$ respectively, we have:-

$$\left\{B_1^{(3,1)}\right\}^T\left\{B_1^{(3,2)}\right\} = \{2 \quad -1 \quad -1 \quad 0 \quad -1 \quad -1 \quad 2 \quad 0 \quad \cdots\}\{2 \quad 1 \quad 1 \quad 0 \quad -1 \quad -1 \quad -2 \quad 0 \quad \cdots\}^T = 0$$

$$\left\{B_1^{(3,1)}\right\}^T\left\{B_2^{(3,2)}\right\} = \{2 \quad -1 \quad -1 \quad 0 \quad -1 \quad -1 \quad 2 \quad 0 \quad \cdots\}\{1 \quad 2 \quad -1 \quad 0 \quad -2 \quad 1 \quad -1 \quad 0 \quad \cdots\}^T = 0$$

showing that $\Phi_1^{(3,1)}$ and $\Phi_1^{(3,2)}$ are orthogonal vectors, as are $\Phi_1^{(3,1)}$ and $\Phi_2^{(3,2)}$, and so forth.

6. Validation of operators

To validate the proposed operators for the further decomposition of subspace $S^{(3)}$ of problems with C_{3v} symmetry, a spring-mass dynamic model with 3 degrees of freedom $\{u_1, u_2, u_3\}$, and having 3 masses and 6 springs interconnected in a C_{3v}-symmetric pattern, was considered. This example was considered in a previous study of the first author [21], where natural frequencies of vibration for all subspaces of the problem were computed, but without the further decomposition of subspace $S^{(3)}$ proposed here. In that previous study, subspace $S^{(3)}$ was indeed shown to have doubly-repeating natural frequencies, but these frequencies were computed using the basis vectors of the 2-dimensional subspace $S^{(3)}$.

In the present validation, operators in Equations (21) and (22) have been applied to the degrees of freedom $\{u_1, u_2, u_3\}$ of the full space of the spring-mass system, resulting in two 1-dimensional subspaces $S^{(3,1)}$ and $S^{(3,2)}$ that separately yield equal eigenvalues (the repeating natural frequencies of subspace $S^{(3)}$) and modes that are orthogonal to each other, thus fully validating the correctness of the proposed operators. Due to space constraints, details cannot be presented here, but will be shown in the oral presentation at the conference.

As additional validation, the buckling of a rigid 3-sided regular polygonal frame, under the compression action of equal joint loads P directed towards the centre of symmetry O, was considered (see Figure 4).

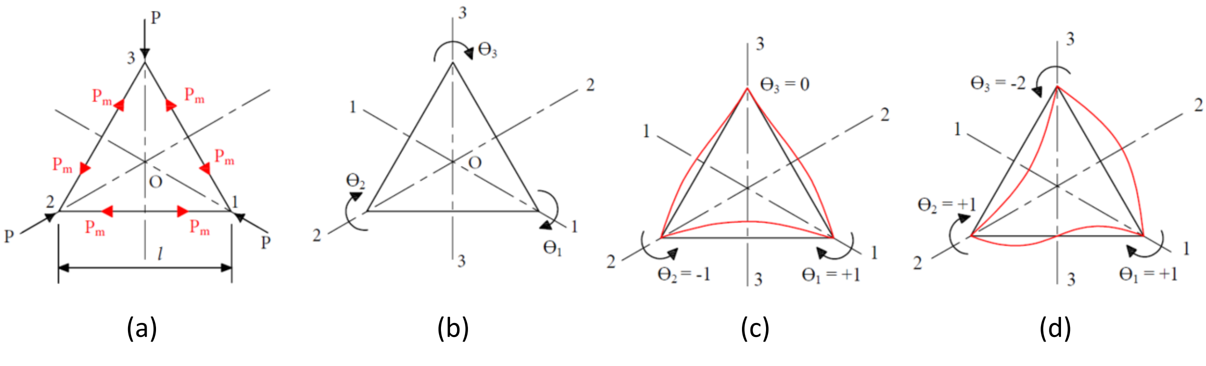

Figure 4: C_{3v}-symmetric triangular frame subjected to point loads P directed towards centre of symmetry: (a) loading configuration; (b) rotational joint freedoms; (c) first mode (C_{1v}-symmetric); second mode (C_1-symmetric) [23].

This example was also considered in a previous study by the authors [23], where buckling eigenvalues for all subspaces of the problem were computed analytically using group theory. In that study, the further decomposition of subspace $S^{(3)}$ was achieved by a search for a linear combination of the two basis vectors of subspace $S^{(3)}$ such that the ensuing basis vectors were orthogonal to each other, thus yielding the basis vectors of the semi-subspaces $S^{(3,1)}$ and $S^{(3,2)}$. In the present work, subspace $S^{(3)}$ of the same frame has been decomposed more systematically using the operators in Equations (21) and (22). This has led to exactly the same results for eigenvalues and mode shapes, confirming the validity of the two operators.

The first two buckling modes have equal eigenvalues, and belong to subspace $S^{(3)}$. Mode 1 (subspace $S^{(3,1)}$) has C_{1v} symmetry (i.e. one axis of *symmetry*), while mode 2 (subspace $S^{(3,2)}$) has C_1 symmetry (i.e. one axis of *antisymmetry*) – see Figures 4(c) and (d). The first eight modes as computed from a finite-element analysis (FEM) are shown in Figure 5, while comparisons between group-theoretical (GRT) and FEM results are shown in Table 1 for the first six modes. The agreement between theoretical and FEM results is excellent.

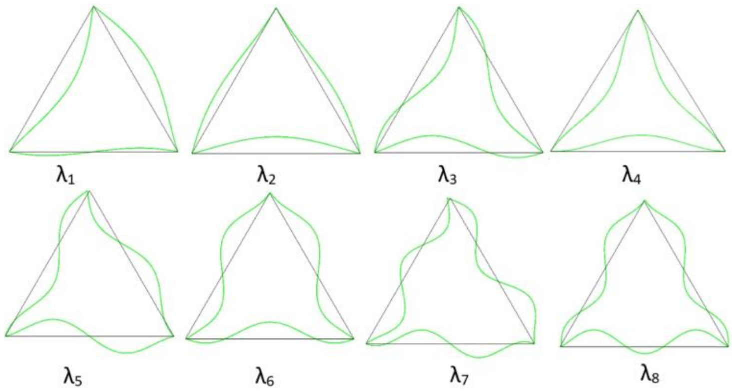

Figure 5: First eight buckling modes of the triangular frame as yielded by FEM [23].

Table 1: Group-theoretic (GRT) versus finite-element analysis (FEM) results for the triangular frame [23] (λ_h denotes the eigenvalue for mode h of the full system).

h	λ_h (GRT)	λ_h (FEM)	% difference in λ_h values	symmetry of mode (GRT)	symmetry of mode (FEM)
1	3.857	3.813	1.2	$C_1\ [S^{(3,2)}]$	C_1
2	3.857	3.813	1.2	$C_{1v}\ [S^{(3,1)}]$	C_{1v}
3	6.283	6.191	1.5	$C_3\ [S^{(2)}]$	C_3
4	6.283	6.191	1.5	$C_{3v}\ [S^{(1)}]$	C_{3v}
5	8.187	8.026	2.0	$C_1\ [S^{(3,2)}]$	C_1
6	8.187	8.026	2.0	$C_{1v}\ [S^{(3,1)}]$	C_{1v}

7. Concluding remarks

In this contribution, we have presented, for the first time, a new pair of operators for the full decomposition of the group-theoretic subspaces of structural configurations belonging to the symmetry group C_{3v}, which describes the symmetry of a 3-sided regular polygon. The results find application in the analysis of symmetric cable nets, space grids, lattice shells and other spatial structures.

Specifically, by acting on the normal variables of a structural problem (such nodal positions and degrees of freedoms), these operators generate two sets of basis vectors that are orthogonal to each other, effectively decomposing the degenerate subspace $S^{(3)}$ (associated with doubly-repeating solutions of the problem) into two independent semi-subspaces $S^{(3,1)}$ and $S^{(3,2)}$. In eigenvalue problems, the two semi-subspaces yield identical sets of eigenvalues. Modes of the same semi-subspace all have the same symmetry type: $S^{(3,1)}$ modes are *symmetric* and $S^{(3,2)}$ modes are *antisymmetric* about a vertical plane of the configuration.

Application of these operators has been illustrated by reference to a double-layer triangular space grid, and their validity confirmed through comparisons with results from the literature. Thus, these operators not only simplify the computation of required quantities, but also "untangle" the symmetries of subspace $S^{(3)}$.

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