Proceedings of the IASS 2024 Symposium
Redefining the Art of Structural Design
August 26-30, 2024, Zurich Switzerland
Philippe Block, Giulia Boller, Catherine DeWolf,
Jacqueline Pauli, Walter Kaufmann (eds.)

Curvature-driven Optimization in Grid-shell design : An innovative Force density method application

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Abstract

This paper introduces an innovative grid-shell design framework by incorporating Willmore energy into the Force Density Method (FDM). The Nonlinear Force Density Method enhances grid-shell form refinement by effectively managing complex constraints and objective functions, building on linear FDM. This approach utilizes the reciprocal discrete Airy stress functions and graphic statics to establish initial grid-shell geometries in equilibrium, with a significant focus on curvature optimization. The Maxwell-Mondrian diagram, combining the form, force, and slope diagrams, plays a central role in calculating discrete Gaussian and mean curvatures crucial for assessing Willmore energy. Proposed method shifts the focus from Gaussian curvature, emphasized from its relation to structural vertical equilibrium, to mean curvature, selected for its relevance in aesthetic considerations. The shift facilitates smoother and more uniform surfaces, essential for both structural efficiency and aesthetic quality in grid-shell designs. By targeting mean curvature optimization, the minimization of Willmore energy provides a distinct and objective framework. This paper details the methodology and demonstrates its applications through examples with square, rectangle and simplified form diagrams inspired by the Great Court Roof of the British Museum.

Keywords: Grid-shell, force density method, curvature-driven optimization, Willmore energy, Mixture of diagrams

1. Introduction

Grid-shell design has advanced through diverse methodologies, notably with graphic statics gaining prominence for its intuitive approach to form and force diagrams[1]. The previous studies have explored many grid-shell design using both 3D Rankine reciprocals[2] and 2D Maxwell reciprocals, the latter being the focus of this paper on 2.5D grid-shell structures. Additionally, the Airy Stress Function (ASF) has been instrumental in studying shells and tension structures, as seen in works like Sehlström's[3]. Numerical methods such as the force density method[4] and dynamic relaxation[5] are well-established. Thrust Network Analysis (TNA)[6], an extension of the FDM, is particularly effective for compression-only structures.

This research focuses on developing grid-shells using the ASF, a potential tool in initial conceptual design stages. Several methods have been developed for constructing ASFs. One method involves creating ASF by directly combining states of self-stresses in an initial mesh and then defining the grid-shell geometry through the force density method[7]. This method demonstrates design freedom, but it also presents challenges in determining the number of independent states of self-stress which is same as the number of possible planar-liftings of the form diagram[8]. Another technique employs a predefined boundary divided into a triangular mesh. By lifting internal nodes based on a pre-determined load path, a polyhedral Airy stress function is formed according to reference [9]. This technique is effective for tension-compression mixed structures but comes with ambiguity in the lifting process and difficulty in controlling edge due to interlinked edges. The third approach uses an electro-magnetic analogy and Biot-Savart law, applicable in situations where no initial mesh is needed, and only a known boundary is present [10]. This approach can create grid-shells with a nearly constant span-to-height ratio and straightforward node lifting. However, advantageous for good engineering properties, it is mainly suitable for compression-only structures.

This paper aims to streamline the initial creation of ASFs, building upon the methodologies of Konstantatou et al.[9] and focuses on simplifying the initial form-setting process to transform perceived design freedom into practical applications. Determining initial z coordinates that satisfy predetermined load paths is a complex task. Automating this selection process introduces a new level of control and efficiency in generating ASFs. Additionally, this paper integrates aesthetic considerations into the structural design process using Willmore energy. The proposed method in this paper allows for a quantitative and approximate assessment of aesthetics, providing a more objective comparison of designs and a pragmatic alternative to intuition-based approaches. In the case of mixed-Airy, ASF and grid-shell have a dual relationship under the same vertical loading conditions[11], implying that determining ASF inherently defines the grid-shell. With various potential ASFs generated through different z coordinates, a multitude of grid-shell designs can be developed. The focus here is on selecting a grid-shell with lower Willmore energy, indicating an optimized balance between structural integrity and aesthetic quality.

The paper is organized as follows: section 2 provides an overview of the background knowledge and key concepts utilized in this research; section 3 introduces the base methodology and its extension using the Maxwell Mondrian diagram with Grasshopper Galapagos and the application of the Nonlinear Force Density method via Python; section 4 presents the results using basic form diagrams, such as square and rectangular shapes, as well as a simplified version inspired by the Great Court Roof of the British Museum form diagrams. Finally, section 5 summarizes the outcomes of this research and discusses potential future works.

2. Background literature

2.1. Graphic statics and Airy stress function

Airy stress function is a useful tool for designing plane-faced and funicular grid-shell. The Airy stress function, $\phi = \phi(x, y)$, is used to describe the stress within the structure through their second derivatives, such as $\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2}$, $\sigma_{xy} = \frac{\partial^2\phi}{\partial x \partial y}$, $\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2}$ [12]. This can be extended to discrete Airy stress function and the folds of plane-faced polyhedron carry all the forces. Therefore, faces have zero stress and the slope between adjacent faces indicates the magnitude of the force[12]. The discrete Airy stress function is a lift of the 2D form diagram. Based on rigidity theory, linearly independent lift of nodes generates linearly independent Airy stress function and as well as linearly independent state of self-stress. This paper only uses triangular meshes, implying every node lifting is linearly independent[12].

Maxwell made a conjecture that if a 2D planar framework is in a state of self-stress, it must be a projection of a plane-faced polyhedron. In addition, Maxwell used the context of projective geometry to construct these diagrams. Based on projective geometry, the pole (α, β, γ) maps to a reciprocal plane $z = \alpha x + \beta y - \gamma$ in 3D space[13]. Therefore, a discrete Airy stress polyhedron, ϕ, has a reciprocal polyhedron, ϕ^* and this can be constructed by mapping a plane in ϕ to a point in ϕ^*. The projection of ϕ^* onto the xy plane is a force diagram. Airy stress function ensures horizontal equilibrium.

2.2. Force density method

The Force Density Method (FDM) has been widely used for form-finding in grid-shell structures. The force density, q, defined as the ratio of axial forces to the lengths of the members, simplifies a potentially nonlinear system of equation into a linear one. The force densities are conserved even during projection, allowing for the determination of the height of the internal nodes z_N from the equation $z_N = D_N^{-1}(p_z - D_F z_F)$. Here p_z represents the vertical loading for internal nodes, $D_N = C_N^T Q C_N$, $D_F = C_N^T Q C_F$where C is the connectivity matrix between nodes and edges, and Q is the diagonal matrix of force densities [5]. This paper utilizes triangular meshes to ensure planarity, addressing a limitation of the FDM, which does not guarantee flatness of faces.

The Nonlinear Force Density Method (NFDM) incorporates additional constraints into the conventional FDM. NFDM treats form-finding as a least-squares problem, seeking to optimize the fit by minimizing the residuals within the system of equations. The iterative optimization process involved in NFDM employs numerical methods such as Gauss-Newton or conjugate gradient methods, which enable the precise determination of form while adhering the specified constraints [5]. Significantly, NFDM is applicable to structures with mixed tension and compression elements.

2.3. Mixture of diagrams

The Maxwell-Minkowski diagram combines two distinct vectors into a single diagram such as form and force diagrams. This diagram depicts the load path and force density, represented by the area and aspect ratio of rectangles, respectively.

A novel concept known as the slope diagram [11], parallels the role of the force diagram, which shares reciprocal relationship with the form diagram through Airy Stress function. In the slope diagram, this mediation is performed by the grid-shell, and the red arrows indicates the inclination of the grid-shell's edges, as shown in Fig. 1. Notably, in plane-faced funicular grid-shells, the edges of both force and slope diagrams are parallel.

The Maxwell-Mondrian diagram synthesizes the form, force, and slope diagram into a unified diagram. The area resulting from integrating the slope diagram is double the mixed area and signifies the vertical load. The mixed area remains constant despite translations of the slope diagram. As shown in Fig. 1, cases 1 and 2 have different slope diagrams, but twice the mixed areas which are represented in blue, are the same. This property allows for the computation of curvature using the mixed area.

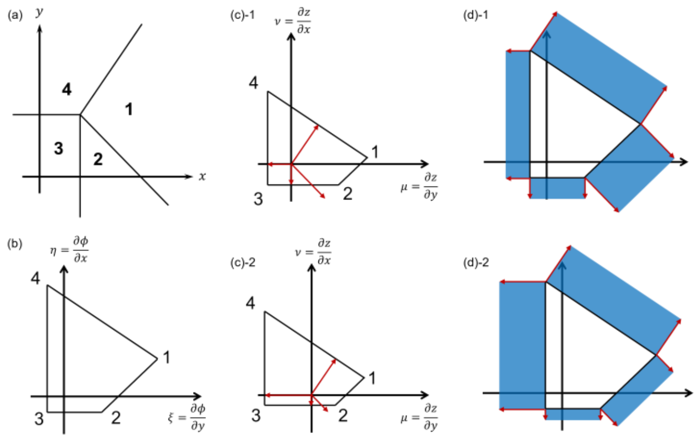

Figure 1: A simple example of (a) form, (b) force, (c) slope and (d) FS (Force & Slope) Minkowski diagrams. The red arrow indicates the slope of the corresponding bar. The blue area in the FS Minkowski diagram represents twice the mixed area[11].

2.4. Isotropic curvature

Isotropic geometry is a fundamental concept underlying the Airy stress function, ϕ, as it directly correlates surface curvatures with stresses. This aspect is particularly beneficial in designing grid-shells subjected to vertical loading. In isotropic geometry, surfaces map (x, y) to $(x, y, f(x, y))$, thus requiring the use of second derivatives of a function, represented here by ϕ [11].

- Isotropic Gaussian curvature: $K = det(\mathbf{H}(\phi)) = \frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial y^2} - \left(\frac{\partial^2\phi}{\partial x \partial y}\right)^2$ indicates local sphericity.
- Isotropic mean curvature: $H = \frac{1}{2}tr(\mathbf{H}(\phi)) = \frac{1}{2}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)$ relates to the surface's bending.

In the discrete version, curvature at a mesh vertex is calculated using area ratios related to the form, force, and slope diagram. The isotropic Gaussian curvature of Airy stress function (K_ϕ) is defined as ${A_i^{*}}' / {A_i}'$ and the mean isotropic curvature of the shell relative to the Airy stress function (H_S^{rel}) as $\widehat{A}_i(z,\phi) / {A_i^{*}}'$ [11].

$$\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - 2\frac{\partial^2\phi}{\partial x\partial y}\frac{\partial^2 z}{\partial x\partial y} + \frac{\partial^2\phi}{\partial y^2}\frac{\partial^2 z}{\partial x^2} = F' \tag{1}$$

Pucher's equation (Eq. 1) establishes a relationship between Gaussian curvature and axial equilibrium in shells under vertical loading. Applying the self-Airy grid-shell condition, Gaussian curvature corresponds to the left side of the equation[11]. This relationship illustrates that a grid-shell's ability to effectively distribute vertical loads, is directly reflected in Gaussian curvature. Incorporating Willmore energy, the integral of mean curvature over a surface, as a measure of deviation from flatness, this paper considers both curvature properties.

2.5. Willmore energy

Energy measures applied to polyhedral meshes often evaluate the smoothness of surfaces. Mesh energy, a comprehensive measure, assesses how accurately a mesh approximates a smooth surface, encompassing curvature, variations in normal vectors, edge lengths, and dihedral angles [14]. The focus is on bending energy, which quantifies resistance to deformation, closely aligning with fairness functionals [15]. These mathematical expressions can evaluate surface smoothness based on curvature properties, defined in terms of mean curvature H and Gaussian curvature κ, expressed as:

$$E_\alpha(\lambda,\theta) = \frac{1}{4}\int(4H^2 - \alpha \times \kappa)dS \tag{2}$$

For $\alpha = 2$, these denotes bending energy. This paper focuses on Willmore energy by setting $\alpha = 0$. While $\alpha = 1$ corresponds to the conformally invariant Willmore energy commonly discussed in computer graphics, an engineering perspective is sufficient to use $\alpha = 0$. This geometric energy captures how much a surface deviates from local sphericity; a surface with lower energy is closer to being spherical. For the discrete version, Willmore energy is applied to a triangulated mesh M, defined as:

$$W(M) = \sum_{e_{pq} \in Edges(M)} 4\left(1 - \cos\gamma\left(e_{pq}\right)\right) \tag{3}$$

In this equation, $\gamma\left(e_{pq}\right)$ is the angle between the normals of the triangles adjacent to edge e_{pq}. When triangles do not degenerate, then $W(M)$ approximates to $\int_M H^2\, dA$ [16].

3. Methodology

The methodology in this research is built upon the work of Konstantatou[9], whose approach to grid-shell design and analysis utilized reciprocal discrete Airy stress functions in conjunction with the Force Density Method (FDM). This framework itself is rooted in the polarities-based approach to graphic statics[13], offering a novel perspective in the structural analysis and design of compression-and-tension structures. While Konstantatou's methodology provided a direct means to achieve static equilibrium without iterative convergence algorithms, this paper extends this methodology by incorporating visual smoothness parameter.

In this paper, the aim is to address the challenge of integrating aesthetic quantification into structural optimization. Specifically, the focus is on calculating Willmore energy, a measure of aesthetics, and using it as an optimization function within the FDM. This approach emphasizes maintaining vertical equilibrium as a primary constraint. By minimizing Willmore energy, the goal is to achieve smoother surfaces, indicative beauty in structural form. Additionally, this paper involves applying secondary constraints to adhere to a pre-established load path and ensuring sufficient curvature to maintain overall structural stability. This section outlines the steps undertaken to incorporate these elements into the extended methodology.

3.1. Explanation of Base Methodology

Konstantatou's approach[9], the polarities-based method constructs four reciprocal diagrams: a form diagram, a force diagram, and their respective Airy stress functions. They are all interlinked and directly created without iteration. Each vertex in the form diagram $F(v, e, f)$ can be lifted to its Airy stress function $P(v, e, f)$, where it corresponds to a reciprocal plane. For example, a vertex v_i in $P(v, e, f)$, represented as $(cA, cB, -C)$, maps to the reciprocal plane π_i' defined as $z = Ax + By + C$, and vice versa. This leads to the creation of the reciprocal Airy stress function $P'(v, e, f)$. By projecting these vertices onto the xy plane, the force diagram $F'(v, e, f)$ is generated. Each closed polygon f_i' in force diagram ensures horizontal static equilibrium at node v_i, and perpendicular edges between diagrams represent Maxwell configuration. The form diagram can be under either self-stress or external loading.

Once the force diagram is given, the force density values are calculated using the Minkowski diagram $MS(F, F')$, where the aspect ratio of each rectangle represents the force density. Then, FDM is applied to determine the z coordinates. In this approach, only triangular meshes are used, which allows for the lifting of internal nodes without limitations. The load path for each edge is determined by the shape of the Airy stress polyhedron at that edge: valleys indicate tension and ridges signify compression. By adjusting the curvature of the Airy stress function, the force in each edge can be controlled. For vertical loads, an approximation of point dead load using the Voronoi area of each node is assumed. Finally, the grid-shell $GS(v, e, f)$, which is lifted from the form diagram $F(v, e, f)$ is created. This grid-shell represents one possible solution of static equilibrium according to lower bound theorem. The variety of available Airy polyhedrons allows for diverse load paths and gird-shells geometries, enhancing design flexibility.

3.2. Objective function 1 from Maxwell Mondrian diagram and Grasshopper Galapagos

In this section, the aforementioned methodology is extended to address the complexities associated with selecting the load path, specifically the difficulty in choosing initial z coordinates of the Airy Stress Function (ASF) that align with a predetermined load path, whether tension or compression. In addition, integrating aesthetic aspects into this framework presents a greater challenge. This approach strives to harmonize the visual smoothness of the ASF with the functional demands of the load path, ensuring that both aesthetic and structural objectives are met.

To identify the best load path, this paper employed Galapagos, a genetic algorithm in Grasshopper. This entails finding the optimized z coordinates for internal nodes. The first step in this process is defining the objective function (f_1), denoted as:

$$f_1 = (\, WillmoreEnergy) + \alpha \times (\, LoadPathComplicancePenalty) \tag{4}$$

In this formula, Willmore energy is used to measure of aesthetic performance with lower values which indicates a smoother surface. The penalty term, with weighting coefficient α ensures adherence to the pre-determined tension and compression paths. This is based on the convexity of each edge, determined by the convergence of normal vectors of the two faces sharing that edge. If this convexity is not maintained during the process, the penalty is applied.

Using Konstantatou's methodology, a grid-shell that ensures vertical equilibrium is first created using linear FDM. This is then extended by using the grid-shell as an Airy Stress function to create a reciprocal grid-shell. From this, a slope diagram is derived by projecting the reciprocal grid-shell onto the xy plane. Informed by Millar's work[11], the slope diagram is then integrated into the Maxwell-Minkowski diagram, resulting in the creation of the Maxwell-Mondrian diagram. Within the Maxwell-Mondrian diagram, the mixed area between the force and slope diagrams at each node of the form diagram is calculated. This mixed area is then divided by the area of the polygon dual to a node in the force diagram to compute the discrete mean curvature at that node. The discrete Willmore energy is calculated by squaring these mean curvature values, multiplying them by the corresponding tributary area at each node.

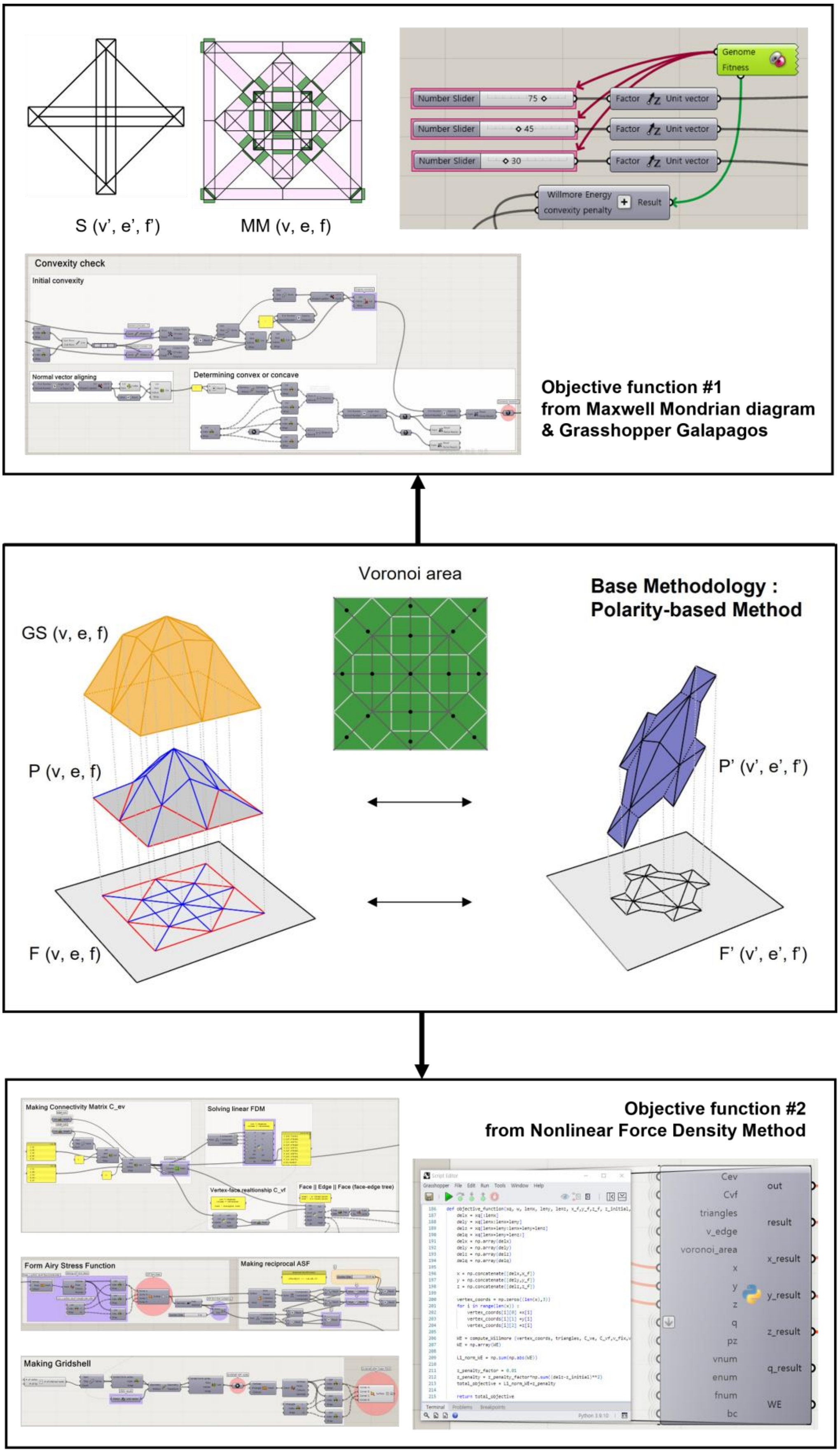

Figure 2: Framework of determining ASFs and designing grid-shells

3.3. Objective function 2 from Nonlinear Force Density Method

In advancing research towards identifying the optimal form, this section employs the Nonlinear Force Density Method (NFDM). This method is more integrated method than the Galapagos approach since the FDM process, constraints, and the solution search algorithm are all combined within one Python script. The objective function in NFDM remains the same as before, which is the Willmore energy. However, the approach to computing Willmore energy in NFDM differs from that used in the Galapagos method. In NFDM, the objective function needs to be expressed in terms of coordinate and force density. Since area calculations using the Maxwell-Mondrian diagram cannot be directly expressed in these terms, the fundamental definition of Willmore energy derived from the geometry of the form is used. Consequently, the computation of Willmore energy solely utilizes the form diagram.

The ‘scipy.optimize.minimize’ function utilizing the SLSQP algorithm, is implemented to determine the form that meets constraints. This algorithm is widely recognized in NFDM for problems involving constraints. It is vital that the number of free internal nodes exceeds the number of constraints. In this segment, constraints play a pivotal role: strong constraints are explicitly incorporated as a part of the constraint function, while weak constraints are applied in the form of penalties. For strong constraints, the vertical equilibrium and the bounds for the x, y, and z coordinates are ensured. Specifically, to preserve the form diagram, changes in x and y coordinates are bounded between [-3,3]. For z coordinates, given assumption that the dead load is proportional to the Voronoi area-unsuitable for steep shells-the z coordinates are limited to maximum of 200, preventing excessive slopes.

During the test simulations to refine this methodology, challenges were faced in achieving convergence. Initial values derived from Galapagos, a genetic algorithm based on global optimization, are integrated to find better starting points and mitigate the risk of local minima. Despite these enhancements, increasing the number of iterations revealed a tendency for the grid-shell to flatten, an effect akin to it becoming a part of a very large sphere. Therefore, a penalty for the large variation in the z coordinate is applied. Consequently, the objective function for the NFDM in this paper for the basic form diagrams is formulated as:

$$\text{minimize} \quad f_2 = (\textit{Willmore Energy}) + \alpha \times (\textit{z coordinate variation penalty}) \tag{5-a}$$

$$\text{subject to} \quad -3 \leq D_N^{-1}\left(p_x - D_F x_F\right) - x_0 \leq 3 \tag{5-b}$$

$$-3 \leq D_N^{-1}\left(p_y - D_F y_F\right) - y_0 \leq 3 \tag{5-c}$$

$$0 \leq D_N^{-1}\left(p_z - D_F z_F\right) \leq 200 \tag{5-d}$$

The objective function for the simplified Great Court Roof form diagrams is similar to that of the basic form diagram, but with slightly different constraints. The summarized overall methodology is demonstrated in Fig. 2.

4. Results

4.1. Objective function 1 from Maxwell Mondrian diagram and Grasshopper Galapagos

Utilizing the Maxwell Mondrian diagram in conjunction with the Grasshopper Galapagos method, diverse optimization results were achieved by setting different load paths. The form diagram, representing the load path, along with the Airy Stress Function, the optimized grid-shell, and the Maxwell Mondrian diagram, are visually expressed in Fig. 3. These illustrations demonstrate the effectiveness of this methodology in controlling the load path, addressing both structural and aesthetic considerations in grid-shell design. Furthermore, it is important to note that the Willmore energy value obtained through the Maxwell Mondrian diagram may significantly vary depending on each load path. This variation is primarily attributable to the scale of the area involved. Since the squared mean curvature, a dimensionless term, is multiplied by the area value, the Willmore energy value becomes highly sensitive to the scale of the area term. The resulting value can differ from the Willmore energy calculated solely based on the form, which is also a dimensionless term. Therefore, the key consideration is ensuring that consistent relative magnitudes of Willmore energy value across different load paths indicate similar trends.

4.2. Objective function 2 from Nonlinear Force Density Method

Integrating all the steps in single Python script and performing Nonlinear Force Density Method presents issues of computational complexity and convergence. Particularly, calculations aimed at maintaining a predetermined load path are time-consuming. To address computational challenges, the Galapagos method was incorporated to provide an initial form for Python-based NFDM input. The strategy is visually documented in Fig. 4. Load path calculations were specifically applied in the Galapagos method and excluded in the NFDM process. Furthermore, the limitation on the change of the z coordinate is justified, as the input form has already undergone optimization, ensuring that the resulting structure does not resemble a flat surface, such as part of a large sphere. As the number of points increases, designing an initially specified load path becomes difficult, and strict adherence to these paths can indeed lead to structurally impractical designs.

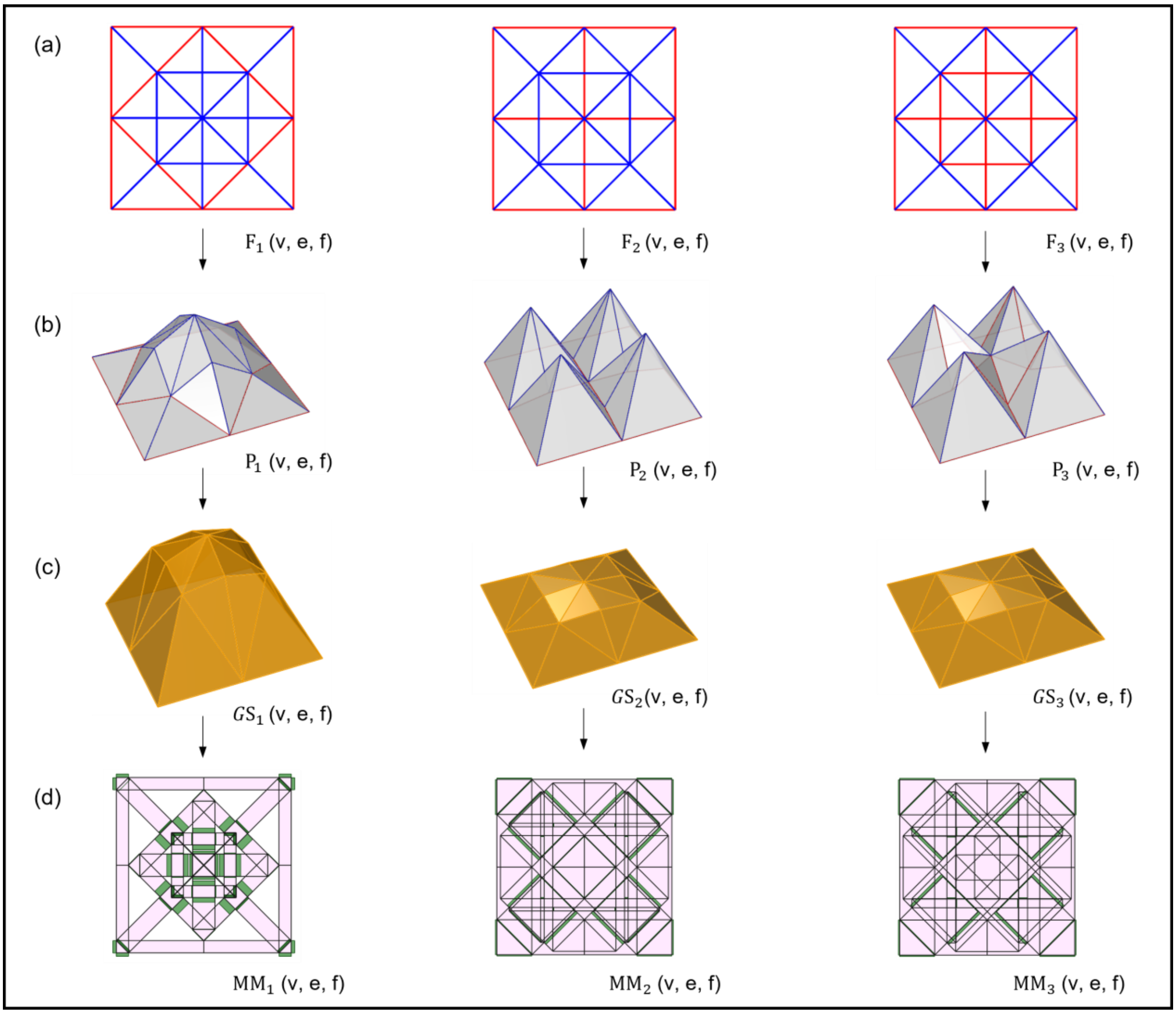

Figure 3: Three different load paths in the same (a) form diagrams $F(v, e, f)$, (b) determined polyhedral ASFs $P(v, e, f)$ by Galapagos, (c) corresponding grid-shells $S(v, e, f)$, (d) Maxwell Mondrian diagrams $MM(v, e, f)$

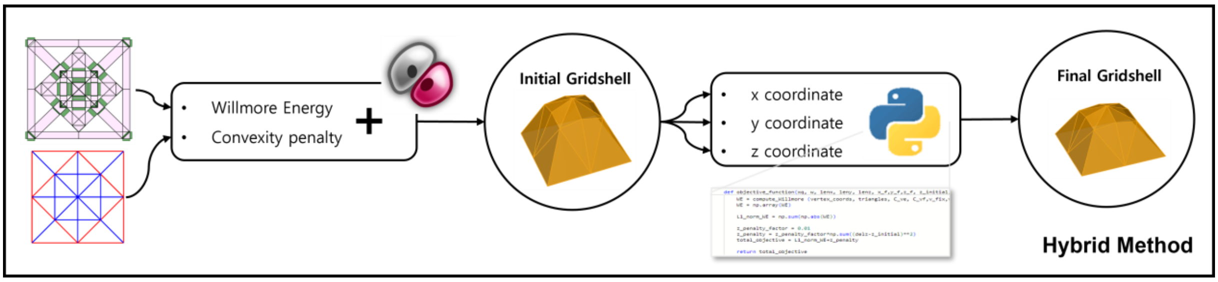

Figure 4: The hybrid method framework combining the Galapagos method and the Nonlinear FDM method.

The application of the hybrid method to a square-shaped form diagram with predetermined load path is depicted in Fig. 5, featuring: (b) corresponding ASF, (c) the initial grid-shell generated by the Galapagos method, and (d) the final grid-shell optimized by the Nonlinear FDM. Willmore energies for each shape are indicated below the image. Fig. 6 and 7 follow a similar format to Fig. 5 but apply to rectangular shapes with different load paths. Fig. 8 shows application to the simplified Great Court Roof form diagrams, while Fig. 9 applies the same method to a different geometry of the same form diagrams. Such observations illustrate a decrease in Willmore energy.

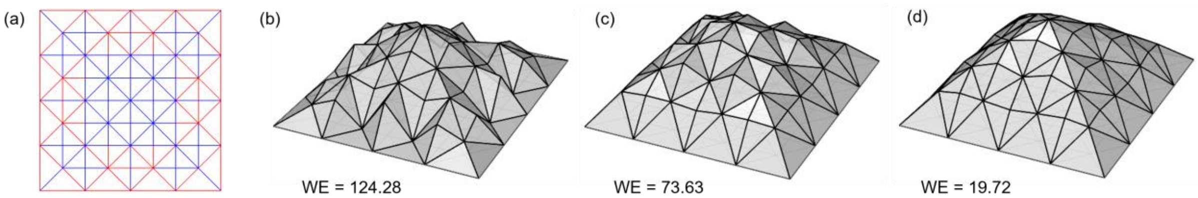

Figure 5: Application of the hybrid method to a square-shaped form diagram

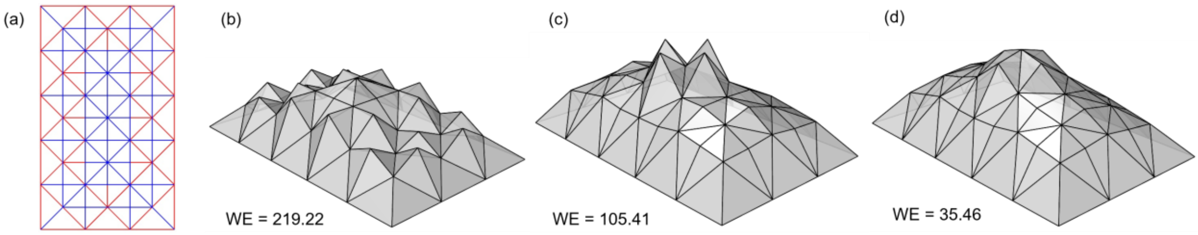

Figure 6: Application of the hybrid method to a rectangular-shaped form diagram for load path case 1

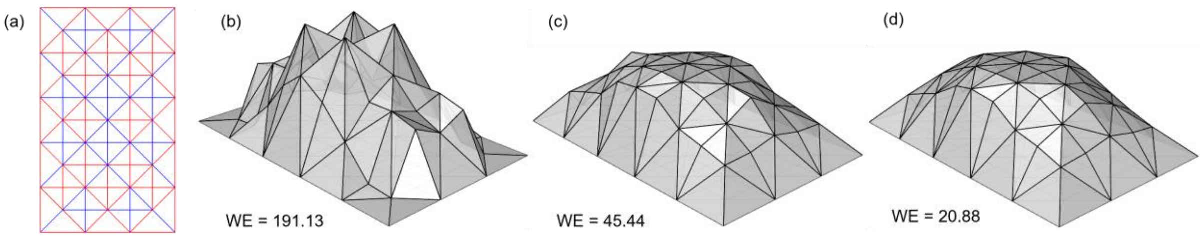

Figure 7: Application of the hybrid method to a rectangular-shaped form diagram for load path case 2

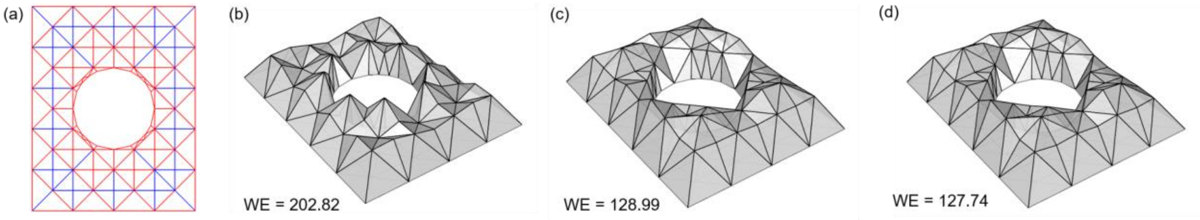

Figure 8: Application of the hybrid method to a simplified version inspired from the Great Court Roof of the British Museum form diagram

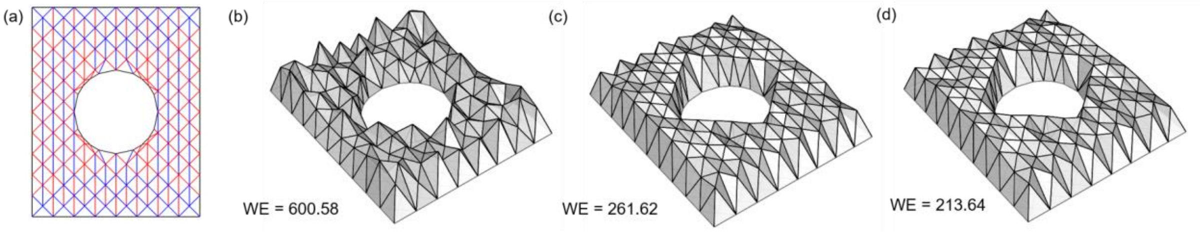

Figure 9: Application of the hybrid method to a simplified version inspired from the Great Court Roof of the British Museum form diagram with different geometry

5. Conclusions

The paper has presented a framework for designing grid-shell that consider both structural and aesthetic aspects. In this approach, Willmore energy is used as a criterion for aesthetic performance and is calculated using two methods: Maxwell Mondrian diagram with the Grasshopper Galapagos method and the Nonlinear Force Density Method. Integrating these methods, the methodology achieves diverse optimization results through various load paths, emphasizing a balance between structural integrity and aesthetic aspect. This outcome assists designers in selecting the initial shape of grid-shells and reduces the complexity and difficulties associated with the design freedom of the Airy stress function.

Applying this approach to different boundary geometries and conditions could provide more insights into its effectiveness. Furthermore, exploring additional measures for assessing aesthetic performance beyond Willmore energy could lead to further developments. The utilization of mixture of diagrams is beneficial for architects to understand structures and integrate them into their designs. Advancing this aspect and more actively employing those diagrams represents a potential area for growth. In this paper, an attempt was made to incorporate buckling problems by calculating Gaussian curvature based on the area ratio. However, due to the excessive computational demands and complexity incorporating this aspect into the Python script, the approach was modified to include a z penalty instead.

Distinguishing between the two method of computing Willmore energy is crucial, particularly in understanding the duality between ASF and grid-shell in mixed-Airy cases. The first method calculates the Willmore energy of the grid-shell. Although this differs from the ASF calculation, it shares a duality relationship with it. The second method, focuses on computing the Willmore energy of the initial ASF input in Python, primarily targeting the ASF that also serves as a grid-shell in vertical equilibrium. The hybrid method begins with an initially optimized grid-shell as the input for Python, ultimately leading to a final optimized grid-shell.

Acknowledgements

This work was supported by the BK21 FOUR (Fostering Outstanding Universities for Research) Project in 2024. (No.4120200113771) The Institute of Engineering Research at Seoul National University provided research facilities for this work.

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