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Layout optimization of vaults: from benchmark to practical solutions

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Abstract

Vaults serve as efficient solutions for long span building structures. Although various form-finding methods (e.g., force density method) are available for designing vaults, the need to predefine the connectivity/mesh at the beginning of the process implies that the resulting designs are unlikely to be minimum material solutions. This paper puts forward a new family of form-finding methods employing numerical layout optimization. The need for predefined mesh patterns is eliminated via a 'ground structure' approach. The formulation is a second-order conic programming problem, thus enabling the identification of theoretical minimum material designs of vaults by employing high-resolution nodal grids. To generate more practical, diverse, and (near-)optimum designs, three preliminary studies are carried out: (a) minimising the number of members; (b) optimising mesh patterns; (c) form alternation via post-processing techniques. To demonstrate the efficacy of the proposed methods, several numerical examples are solved.

Keywords: vault, form-finding, layout optimization, minimum material design

1. Introduction

Vaults serve as efficient solutions for long span buildings structures. The forms of vaults are of significance, as they can be chosen to eliminate bending for maximum structural efficiency. Earlier works in form-finding can be traced back to the physical-based hanging chain (Antoni Gaudí), soap film (Frei Otto) and hanging cloths (Heinz Isler) models in designing funicular structures carrying only compressive forces. Numerical form-finding methods were later developed, the classical approaches can be categorized in three main families (Adriaenssens et al. [1], Veenendaal and Block [2]): stiffness matrix methods (Sive and Eidelman [3]); Geometric stiffness methods such as the force density method (Schek [4]) and thrust network analysis (Block and Ochsendorf [5]); dynamic equilibrium methods including dynamic relaxation (Barnes 1977[6]) and particle-spring system (Kilian and Ochsendorf [7]). Despite the wide range of methods available, they often require predefined connectivity or mesh to function effectively, a crucial aspect that will inevitably have potentially significant impacts on the structural efficiency of the outcome forms. Furthermore, the optimality status of the solutions generated by these methods remains undiscovered, as none of them are capable of identifying or predicting the theoretical minimum material designs. Particularly given the climate crisis, and the urgent need to reduce embodied carbon associated with building structures, it is important to obtain theoretical minimum material designs, which can then be utilized as benchmarks to evaluate the structural efficiency of the designs obtained via classical form-finding methods.

To obtain the theoretical minimum material designs of vaults, here a new family of form-finding methods employing numerical layout optimization (Dorn et al. [8], He et al. [9]) is put forward. The method has previously been applied to form-finding problems for arch structures, using the transmissible load concept (Darwich et al. [10]). However, the solutions obtained may comprise multiple surfaces; furthermore, the use of 3D design spaces increases computational cost. Jiang [11] introduces a reduced ground structure with an aim to improving computational efficiency, albeit at the expense of sacrificing its mathematical rigour of attaining theoretically minimum solutions due to the heuristics involved. Recently, a theoretical breakthrough was made by Bołbotowski [12], who reformulated the form-finding problem as a second order conic programming (SOCP) problem involving 2D design spaces, which was later articulated by He et al [13, 14]. The reformulated optimization problem is convex, so globally optimal solutions are guaranteed, which means that the resulting vaults are minimum material designs, with maximum stiffness, for a given 'ground structure'. High-precision numerical benchmarks for theoretically minimum material designs can therefore be generated by employing high-resolution nodal grids.

The resulting vault designs generated via layout optimization often exhibit complex structural forms resembling the Michell continua (Michell [15]), for this reason, it has primarily been utilized as benchmarks rather than being applied in more practical designs. This paper therefore explores several simplification techniques to enhance the practicality of vault layout optimization, involving an extension of the formulation to control specific features of the structure, including the total number of members and mesh patterns. This results in a mixed-integer SOCP problem, ensuring global optimum solutions, albeit with an associated escalation in computational costs. Furthermore, a post-processing step is employed to alter the vault designs through mesh refinement and re-optimization. Given its heuristic nature, it yields only local optimum solutions. The simplification and post-processing approaches offer a means of generating a gallery of vault designs by experimenting with a range of input parameters, while their structural efficiency is assessed using benchmark solutions provided by standard layout optimization techniques.

The paper is organised as follows. In section two, the standard vault layout optimization procedure is briefly outlined. In section three, the proposed simplification and post-processing techniques are described. In section four, numerical examples are solved to demonstrate the efficacy of the methods. Finally, in section five, key conclusions are drawn.

2. Vault layout optimization

The layout optimization process for vaults involves a number of steps, as shown in Figure 1. Firstly the design domain, load and support conditions are specified, Figure 1(a). Secondly, the design domain is discretized using nodes, Figure 1(b). Thirdly, these nodes are interconnected with potential members to create a 'ground structure', Figure 1(c). Finally, the optimal structural layout and nodal elevations are identified by solving a second order cone programming (SOCP) problem, Figure 1(d), which is formulated as:

$$Prob.\ 1:$$

$$\min_{q_{xy}, q_z, r} \quad V = \frac{1}{\sigma}\mathbf{l}_{xy}^{\mathrm{T}}\left(\mathbf{q}_{xy} + 2\mathbf{r}\right) \tag{1a}$$

$$\text{s.t.} \quad \mathbf{B}_{xy}\mathbf{q}_{xy} = \mathbf{0} \tag{1b}$$

$$\mathbf{B}_z\mathbf{q}_z = \mathbf{f}_z \tag{1c}$$

$$2rq_{xy} \geq q_z^2, \quad \text{for all members,} \tag{1d}$$

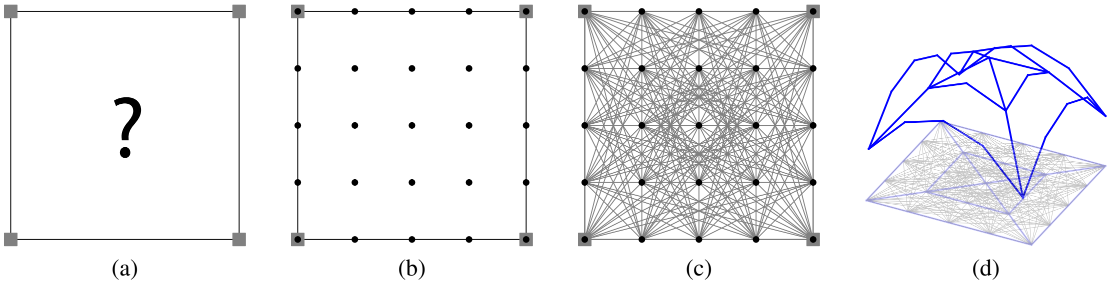


Figure 1: Vault layout optimization steps: (a) specify design domain, loads and supports; (b) discretize domain using nodes; (c) create the 'ground structure' by interconnecting nodes with potential members; (d) use optimization to identify the optimal vault layout and nodal elevations

where, V is the total structural volume; σ is the limiting compressive stress; $\mathbf{l}_{xy}$ is a $m \times 1$ vector of projected member lengths on the xy plane, where m is the number of members. $\mathbf{q}_{xy}$ and $\mathbf{q}_z$ are $m \times 1$ vectors represent, respectively, the resolved member internal forces on the xy plane and z axis, as shown in Figure 2(b); $\mathbf{B}_{xy}$ is a $2n \times m$ equilibrium matrix on the xy plane and $\mathbf{B}_z$ is a $n \times m$ equilibrium matrices in the z axis, where n is the number of nodes. r is an auxiliary variable introduced to simplify the optimization problem, interested readers can refer to Bołbotowski [12] and He et al. [14] for details. Note that design variable l_z shown in Figure 2(a) is not explicitly included in the formulation (1). Nevertheless, it can be calculated when the optimization is solved, using $l_z/l_{xy} = q_z/q_{xy}$. Then the optimum nodal elevations can be obtained.

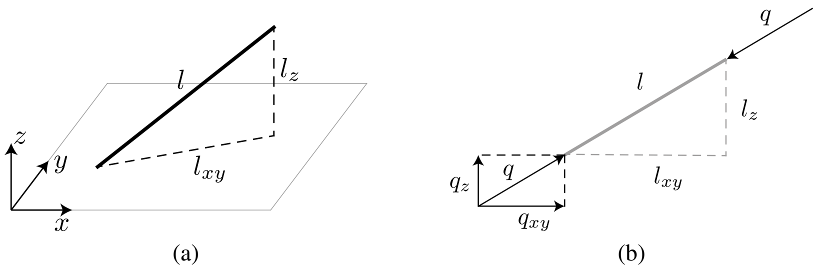


Figure 2: Variables of a single member: design variable l_z, and state variables q_{xy}, q_z

Formulation (1) is a SOCP problem, therefore the global optimality of the solution obtained for a given 'ground structure' (e.g., Figure 1(c)) is mathematically guaranteed (i.e., the design consuming the least amount of material is found). Unlike classical form-finding methods, here an initial mesh or connectivity is not required; instead, the layout optimization process will identify the optimum member layouts from the 'ground structure' - a significant increase of design freedom compared with the classical methods. The solutions of the layout optimization will converge to the theoretical minimum material design when finer nodal grids are utilized. Figure 3 illustrates a convergence study utilizing an extrapolation scheme (see Darwich et al. [10], Bołbotowski et al. [16]), where highly accurate numerical solutions are obtained.

It can be observed from Figure 3 that layout optimization solutions with a high resolution nodal grid resemble the so-called Michell continua (Michell [15]), formed with infinitely small members. The solutions provide theoretical minimum material designs, which can therefore be used as benchmarks to evaluate the structural efficiency of designs generated via other methods.

While the standard layout optimization method produces efficient vault designs, the solutions often

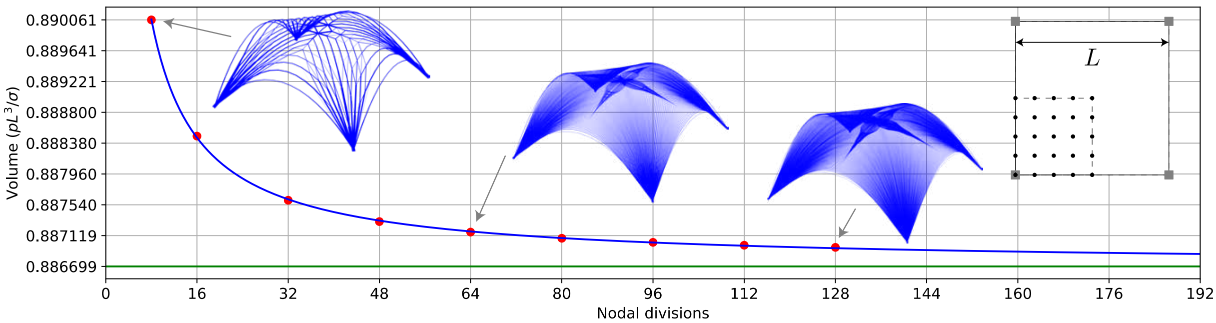


Figure 3: Square domain under pressure load p (He et al [14]): optimum volume v.s. nodal refinement (nodal divisions considered within a quarter domain due to symmetry)

exhibit highly complex forms. To generate more practical, diverse, and (near-)optimum designs, simplification and post-processing techniques are introduced below.

3. Simplification techniques

3.1. Simplification by reducing the number of members

A simple approach to improve practicality in layout optimization is to use mixed integer programming (MIP), where each member is associated with a Boolean variable indicating its existence (see, e.g, Fairclough and Gilbert [17]):

$$q_{zi} \leq t_i \delta_{\text{max}}, \quad \text{for } i \in \{1, 2, ..., m\}, \tag{2}$$

where $t_i \in \{0, 1\}$ is a Boolean variable, δ_{max} is a sufficiently large positive constant. Therefore, the total number of members can be expressed as:

$$M = \sum_{i=1}^{m} t_i \tag{3}$$

where M is the total number of members, which can be used as the new objective function:

$$\begin{aligned} &Prob.\ 2: \\ &\min \quad M && \text{(4a)} \\ &\text{s.t.} \quad V \leq V_{\text{ref}} && \text{(4b)} \\ &\qquad \text{Constraints (1b), (1c), (1d) and (2),} && \text{(4c)} \end{aligned}$$

where V_{ref} is a reference volume.

3.2. Simplification by identifying optimum mesh patterns

Instead of employing a fully connected 'ground structure', another practical approach is to formulate a mesh optimization problem. This is achieved by restricting the 'ground structure' to consist only of short members and subsequently determining the optimal mesh patterns associated with it. An example is shown in Figure 4. A level 1 connectivity (Figure 4(b)) has two mesh variations (Figure 4(c)); and the design freedom is increased (i.e., more mesh variations) when the connectivity level is increased, see Figure 4(d) and (e). To identify the optimum mesh pattern, the following constraint is considered:

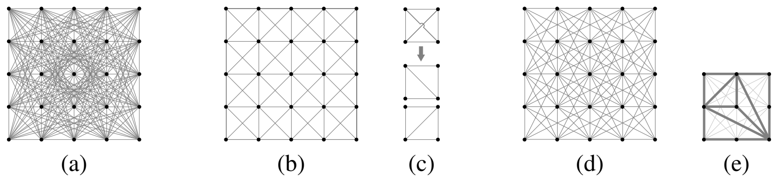


Figure 4: Varying connectivity in the 'ground structure': (a) fully connected; (b) and (c), level 1 connectivity and two potential mesh patterns; (d) and (e), level 2 connectivity and one potential mesh patten

$$\mathbf{P}\mathbf{t} \leq \mathbf{p}_{\text{def}}, \tag{5}$$

where $\mathbf{P}$ is a mesh connectivity matrix specifying the all connected members in each mesh pattern, $\mathbf{t} = [t_1, t_2, ..., t_m]^{\mathrm{T}}$ is a vector containing Boolean variables associated with each member; $\mathbf{p}_{\text{def}}$ is a mesh definition array specifying the number of members required to form each mesh configuration. Constraint (5) can be used in conjunction with $Prob.$ 1 or 2, for example:

$$Prob.\ 3:$$

$$\min \quad V \tag{6a}$$

$$\text{s.t.} \quad \text{Constraints (1b), (1c), (1d), (2) and (5),} \tag{6b}$$

which identifies an optimum mesh pattern for the minimum material design.

3.3. Form alternation via post-processing steps

The optimal layout or mesh identified through layout optimization can be utilized as initial input for classical form-finding methods, leading to a post-processing step that can generate various forms contributing to a diverse design gallery. For sake of simplicity, here the force density method (FDM) is employed. The main steps are as follows:

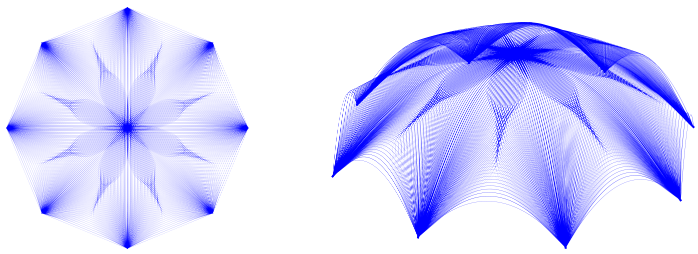
1. Obtain a benchmark solution by solving the standard vault layout optimization problem ($Prob.$ 1) employing a fine nodal grid in the 'ground structure'.
2. Generate a (near-)optimum vault structure by solving any of the extended vault layout optimization problems ($Prob.$ 2 or 3).
3. Obtain an initial connectivity/mesh for FDM by extracting members with non-zero area from the (near-)optimum vault.
4. Assign force density to all members, considering one of the following scenarios:
   - each member has a force density identical to that of the (near-)optimum vault.
   - assign a unit force density to all members.
   - members are firstly grouped using e.g., K-means clustering algorithm, based on their areas in the (near-)optimum vault; then each group is assigned an average force density.
5. Optimize the force density in each scenario by solving a linear search problem: $\min_{\alpha} V(\alpha \mathbf{f}_{\text{d}})$, where α is a multiplier to be optimized, $\mathbf{f}_{\text{d}}$ is the force density vector obtained in the previous step.
6. Evaluate structural efficiency of all designs using the benchmark in step 1.

4. Numerical examples

To demonstrate the efficacy of the methods, several examples are solved.

4.1. Eight-sided vault

The example considers an eight-sided domain with pin supports at all corners. A benchmark solution is obtained by solving $Prob.$ 1 using a relatively fine nodal grid. Taking advantage of symmetrical conditions, only one-eighth of the domain is modelled, resulting in 1,326 nodes and 878,475 potential members in the 'ground structure'. The reported optimum volume is $V_0 = 6.63pL^3/\sigma$ (Figure 5), where L is the length of domain edge, p is the magnitude of the area load.

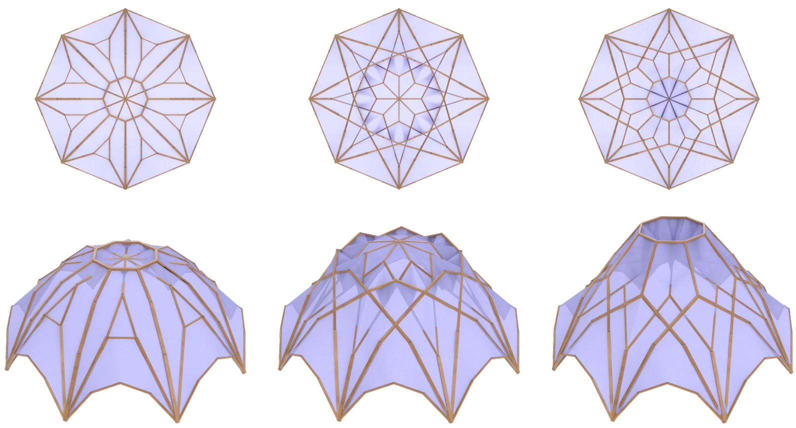


(a) (b)

Figure 5: Eight-sided vault, benchmark solution, $V = 6.63pL^3/\sigma$: (a) top view of optimum vault layout; (b) 3D view

4.2. Minimizing number of members

$Prob.$ 2 is solved to generate simpler designs by minimizing the number of members subject to small volume increase. Note that since integer programming is computationally expensive (as shown in various studies, e.g., in Fairclough and Gilbert [17], Nanayakkara et al. [18]), here only a coarse nodal grid is considered. The solutions are shown in Figure 6. It can be observed that the designs exhibit significant variation as the volume increases. Furthermore, the solutions are no longer single-layered, indicating an area for improvement in future work.



(a) (b) (c)

Figure 6: Eight-sided vault, minimizing number of members by solving $Prob.$ 2 subject to small volume increase ratio: (a) 1%; (b) 5% ; (c) 10%

4.3. Identifying optimum mesh patterns

Similarly, $Prob.$ 3 is solved to identify optimum mesh patterns.The outcome structures obtained using level 1 and 2 connectivities remain similar, see Figure 7(a) and (b), although the former is more computationally efficient than the latter. Therefore, a finer nodal grid is employed in Figure 7(c), resulting in a design that is slightly more efficient but also more complex.

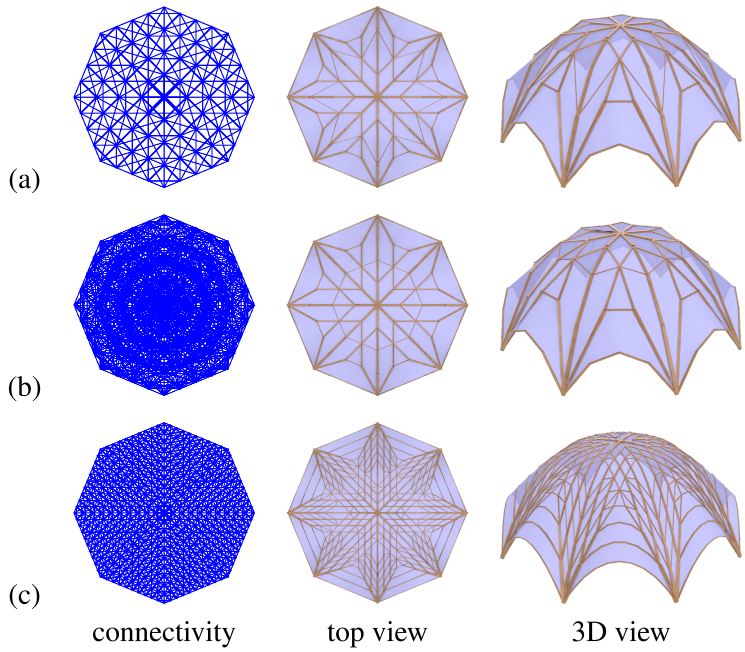


Figure 7: Eight-sided vault, identifying mesh patterns by solving $Prob.$ 3. (a) level 1 connectivity, $V = 1.003V_0$; (b) level 2 connectivity, $V = 1.003V_0$; (c) level 1 connectivity with a finer nodal grid, $V = 1.002V_0$

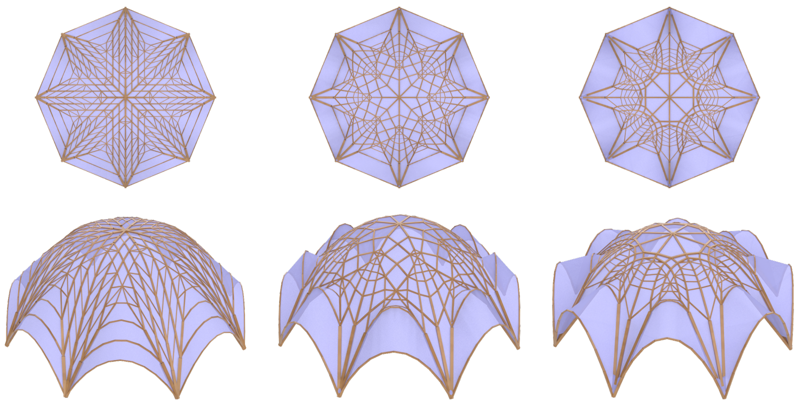
4.4. Post-processing step

The post-processing technique described in Section 3.3. is utilized to generate diverse designs. The solutions are shown in Figure 8. In (a), each member has force density identical to that of the (near-)optimum vault; therefore the resulting layout closely resemble the initial design. In (b), a unit force density is assigned to all members, leading to a significantly different form. In (c), members are categorized into three groups via the K-means clustering algorithm; each group of members shares an averaged force density. This generates a slightly more efficient design than (b). It is worth noting that, despite their varying forms, all the results remain structurally efficient designs, exhibiting only minor deviations compared to the benchmark.

5. Conclusion

In this paper, a new family of form-finding methods employing numerical layout optimization is introduced. The standard vault layout optimization is extended to generate more practical and diverse (near-)optimum designs. The main conclusions are as follows:

- Vault layout optimization is a highly efficient form-finding method that is capable of identifying theoretical minimum material designs of vaults.



(a) (b) (c)

Figure 8: Eight-sided vault, post-processing via FDM: (a) each member has force density identical to that of the (near-)optimum vault, $V = 1.011V_0$; (b) all members have the same force density, $V = 1.028V_0$; (c) using three member groups, $V = 1.026V_0$

- Preliminary studies show that diverse (near-)optimum vault designs can be generated by (a) minimising the number of members; (b) optimising mesh patterns; (c) post-processing the optimum layout via classical form-finding methods.

Acknowledgments

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