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Spanning Trees and Half-Edge Mesh construction in the complete Graphic Statics method for frames

Georgios-Spyridon ATHANASOPOULOS*, Yankun YANG[a],
Russell FEATHERS[b], Allan MCROBIE[c]

*Thornton Tomasetti CORE Studio
12-16 Clerkenwell Road, London EC1M 5PQ
GAthanasopoulos@thorntontomasetti.com

[a] HOK
[b] TikTok
[c] University of Cambridge, Department of Engineering

Abstract

The development of geometric methods that enhance the understanding of structural principles is under the focus of Graphic Statics. The reciprocity between a diagram of a form of a given topology and its diagram of force is of particular significance to represent a structure in state of self-stress. Following the thread from Maxwell and Rankine and the Airy stress functions, late developments have aimed to generalize the approach to address Rankine incompleteness in more generic topologies and to expand the methods to include the geometric construction of moments for the study of frames with the Corsican Sum (or generalized Minkowski Sum) and the complete Graphic Statics for self-stressed frames. Apart from attempts to apply the theory in the design of moment-resisting structures by introducing an architectural hypothesis framework, a clearer visual representation of it has not yet been shown. This research aims to provide a simplified depiction of the main complete Graphic Statics for self-stressed frames doctrine, which is based on the oriented four-dimensional loop concept where bivector components represent a general stress resultant. A primary objective is to render the intuitive aspects of the intricate methodology deriving from the concepts of algebraic topology. Developments are made with the integration of computational tools directly related to the theory, such as Spanning Tree algorithms, and with the novel introduction of the application of the Half-Edge Mesh (HEM) data structure as an alternative construction method of the topology. Applications include the trivial use of the directed and undirected Minimum Spanning Tree (MST) in two-dimensional generic and rectilinear layouts, as well as the Half-Edge Mesh construction approach applied on the same topology. A two-and-a-half-dimensional approach for the HEM method is also discussed and a critical assessment is made to compare results from a computational design perspective.

Keywords: Graphic Statics, Maxwell, Rankine, Reciprocal diagrams, Form and force diagram, Polyhedral geometry, Graph Theory, Spanning Tree, Minimum Spanning Tree, Half-Edge Mesh, Frames, Moment frames

1. Introduction

Geometric methods in structural design are crucial to intuitively grasp form and force reciprocity and are the focus of Graphic Statics. Reciprocal form and force diagrams in the case of 2D and 3D trusses in state of self-stress were thoroughly studied by Maxwell [1] and Rankine [2] in line with the focus on polyhedral Airy stress functions. Latest developments [3][4][5] address Rankine incompleteness to include more generic topologies beyond trusses together with the introduction of shear force and moments with the use of Clifford Algebra. Advancements [6] have led to a complete Graphic Statics method for frames with the combination of elementary homology theory and Grassmann Algebra.

Notably, applications have also extended to the study of kinematics for trusses [7] and frames [8] with Williot-Mohr and Muller-Breslau [9] methods and architectural framework applications for the complete method have been introduced [10][11].

Figure 1 illustrates a typical example of a 2D truss in state of self-stress and its reciprocal force diagram formation from its polyhedral stress function. The planar diagrams demonstrate that the same sum of form and force diagram can be constructed by trivially shifting form diagram closed 'loop-like' pieces and tracing their motion to create areas that then represent the forces along the bars. Depending on whether the motion is perpendicular to a bar or not, force can be axial or shear -which would then produce moments (both bending and torsional). Along these lines, the complete Graphic Statics for frames bases the construction of the 'loop-like' pieces on the drawing of a Spanning Tree along the input topology's nodes. This research further focuses on the Spanning tree types that can be used and aims to advance the method with the novel introduction of a Half-Edge Mesh [12] as an input topology.

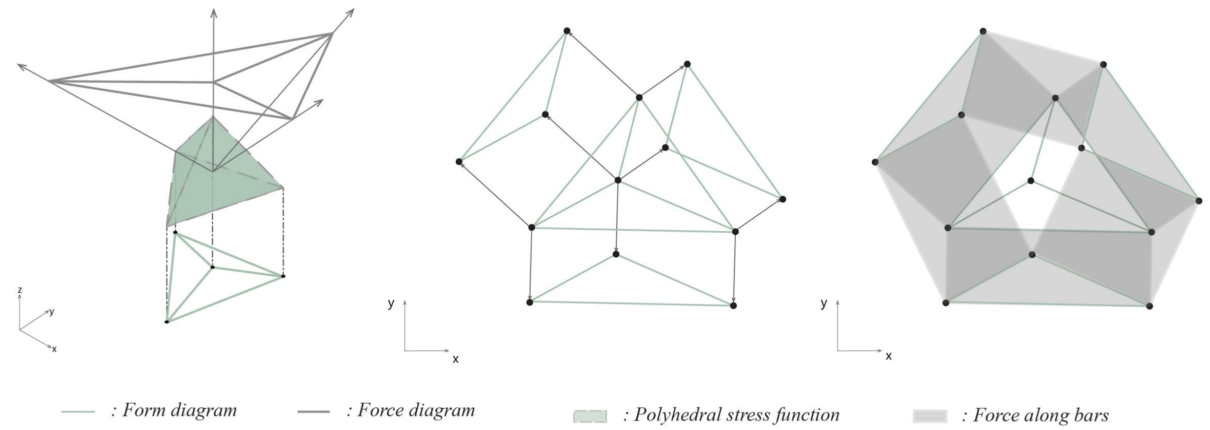

Fig. 1: A typical 2D truss in state of self-stress with its polyhedral stress function whose normal vectors construct the force diagram. These can be also created by simply shifting form pieces onto the structure's plane.

2. Background

2.1. The complete Graphic Statics method for frames

A simple example of the complete Graphic Statics method for frames is shown in figure 2 without the moment representation (for clarity). An initial topology (the main Graph) is used to create a Spanning Tree (in this case, undirected). The Graph edges not in the Spanning Tree are creating a shortest walk path using the edges of the Spanning Tree and form a closed Graph cycle, or loop. The trivial shifting of the loops onto the plane will produce force diagrams (axial and shear). Additionally, a set of cross products of the distance from a unique point for each loop to its endpoints multiplied by the force vector construct the moment (sum of bending and torsion) vectors. The complete axial-and-shear and moment diagram of the structure is then the sum of the total force and moment vectors running along the unique segments of the initial topology. Before proceeding with the force and moment representation it becomes apparent that the articulation of the loops clearly determines the final diagram's complexity which follows the original formation of the Spanning Tree. This loops articulation is the primary focus of this research as types of Spanning have not been examined so far.

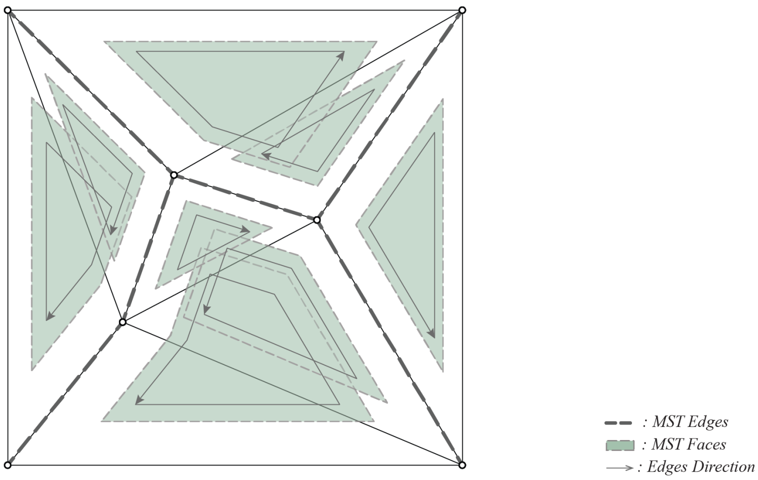

Fig. 2: Complete Graphic Statics for frames, a Minimum Spanning Tree (MST) creates Graph cycles (loops) from a given topology. These can then be shifted to design forces and moments.

2.2. The Half-Edge Mesh (HEM) construction method

Information storage and retrieval are important factors that drive polygon mesh construction methods. A typical representation uses a shared list of vertices and a list of faces with its corresponding vertices. This approach lacks more adjacency information (edges to faces, edges to vertices, etc.) that can be typically required for editing and would demand further time-consuming processes to query. The half-edge data structure is a more advanced mesh representation that allows for more information to be inherently stored and thus, to be retrieved faster. In this type of structure, mesh edges are replaced by half-edge pairs of opposite directions that are positioned circularly along a mesh's face boundary. Together with its pair information, a half edge also stores its vertex information and its corresponding face. Figure 3 illustrates a typical half edge polygon mesh. Face boundaries follow orientations based on the opposite directions of the half edges.

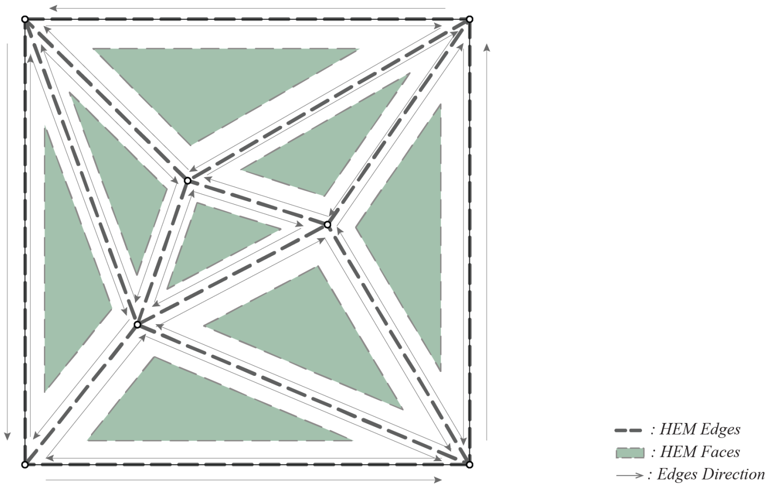

Fig. 3: Half-Edge Mesh construction method (HEM), the edges of the mesh construct the Graph cycles (loops) for the Complete Graphic Statics method.

3. Applications in the complete Graphic Statics method

3.1. Directed Minimum Spanning Tree

The following example uses the Complete method for frames to create force and moment diagrams with the use of a directed Minimum Spanning Tree. Based on a given topology, the directed MST defines a series of graph cycles drawn by the endpoints of the members not on the MST creating a closed path (or loop) using the shortest walk along the directed MST. The cycles are then translated into space (here onto the *xy* plane) to create the force reciprocal diagram. Any moving vectors can be selected to visualise the resultant of axial and shear force along the bar. It is then the cross product of the force along each loop multiplied by the lever arm distance from a reference point that constructs the moment vector on each of the cycle's endpoints for each loop. Each bar's force is the sum of its corresponding cycles' segments that it belongs to. Same applies for moment vectors. In our case, the centroid of each loop is selected as a start of the $\vec{r}$ vector. The moment vector constitutes both of a torsional moment and bending depending on the initial force direction. Total sum of moment vectors manifests moment equilibrium as it equals to 0, and same applies for the force vectors along the bars. It becomes apparent that the total number of additions will depend on the total number of cycles created by a particular topology. However, it should be noted that a directed Spanning Tree calculation would require a time-consuming algorithm by itself.

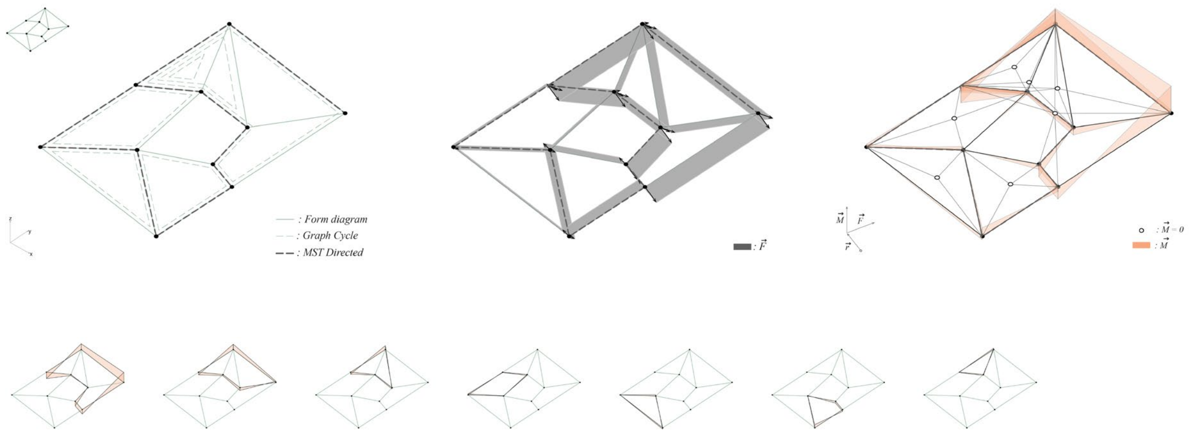

Fig. 4: Directed Minimum Spanning Tree, the Graph cycles (loops) are shifted onto the plane to construct force diagrams, lever-arms from each loop's centroid are drawn to construct the moment vectors along the loop. Total force and moment along a member are the sum of its corresponding loops' segment forces and moments.

3.2. Undirected Minimum Spanning Tree

Figure 5 demonstrates the exact same procedure for the case of an undirected MST, which can be done with less expensive algorithms. The initial topology is treated as the input graph to compute the MST and then the members of the structure not on the MST are used to construct graph cycles similarly as above. The shifting of the loops onto the plane determines the amount of force to be carried along each bar via the summation of its corresponding loop segment forces. Cross products of lever arms from the loops' centroids multiplied by each cycle's force give the moment vectors to be added. Total sum of moment vectors equals to 0 to manifest moment equilibrium and same applies for the force. Compared to the previous solution, apart from a less expensive computational algorithm for the MST, this option can lead to a less complex configuration for summing up force and moment vectors as it may minimize the number of overlapping loops. That also provides a more intuitive understanding of the method that can make the force design clearer as it is arbitrary drive from the manoeuvres. Lastly, it should be noted that the number of edges on an undirected MST for the case of a 2D graph is always *E= V-1*, where *V* the number of vertices according to Euler's formula, which can be useful.

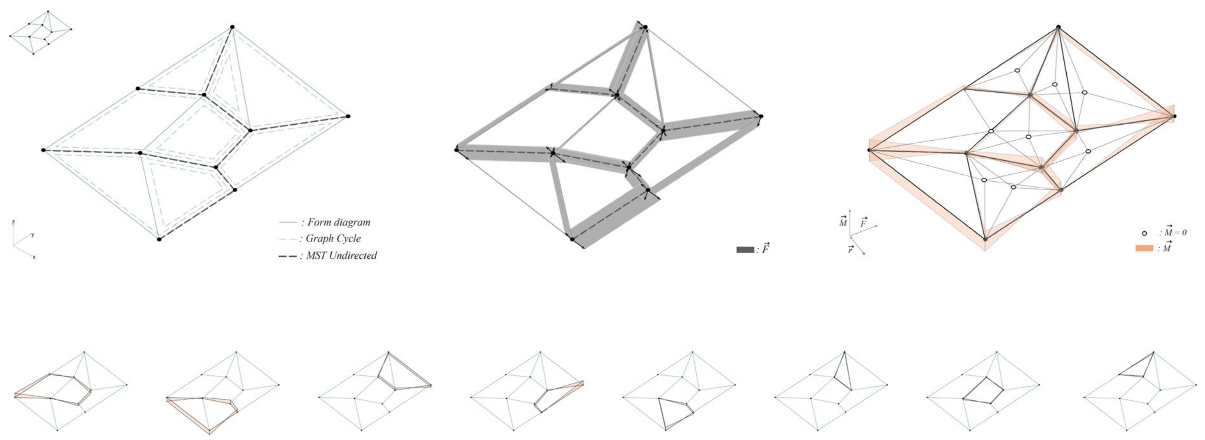

Fig. 5: Undirected Minimum Spanning Tree, the Graph cycles (loops) are shifted onto the plane to construct force diagrams, lever-arms from each loop's centroid are drawn to construct the moment vectors along the loop. Total force and moment along a member are the sum of its corresponding loops' segment forces and moments.

3. The Half-Edge Mesh construction method

The application of the HEM construction method on the previous example is shown on figure 6. The initial topology is now used as input edges for a mesh. Mesh faces are now turned into loops and the half-edges replace the MST. Clearly, the application shows that making use of the HEM data structure can lead to more trivial solutions for the Complete Graphic Statics method for frames. Saving computational time with HEM over MSTs can allow for inputting more complex topologies. Furthermore, this time elimination can allow for real-time manipulation of the initial topology with instant force and moment calculation without intermediate steps. The HEM configuration offers a set of cycles easier to manoeuvre due to their lower number of adjacencies and overlaps, which can be particularly useful for force design. Most importantly though, it provides a by-default initial orientation of the loops that can be directly used.

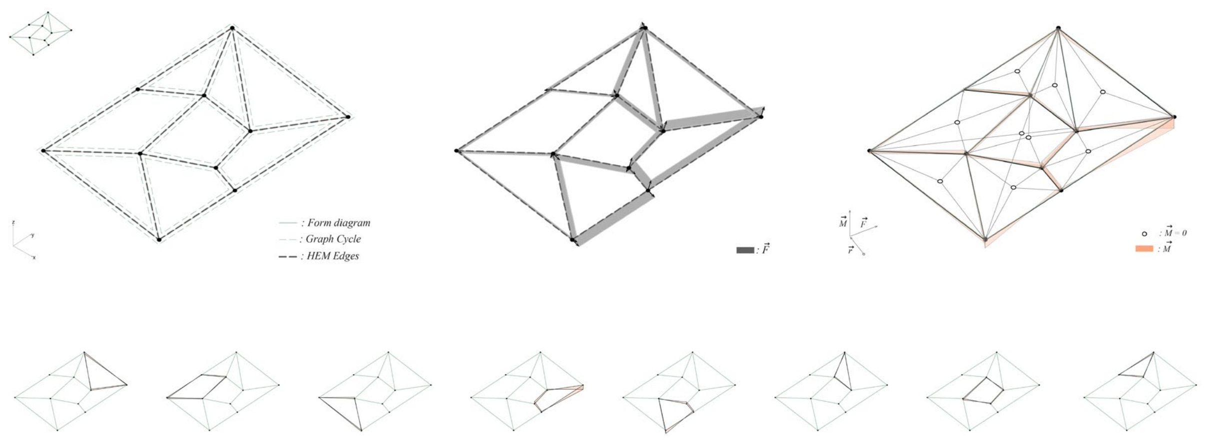

Fig. 6: Half-Edge Mesh, the Graph cycles (loops) are shifted onto the plane to construct force diagrams, lever-arms from each loop's centroid are drawn to construct the moment vectors along the loop. Total force and moment along a member are the sum of its corresponding loops' segment forces and moments.

Further to the example above, the following figure shows that same principles also apply to rectilinear layouts. This time, the Minimum Spanning Trees are turned into rectilinear Spanning Trees by following shortest walks along the exiting topology. The figure illustrates a comparison between all three options discussed. Clearly, the HEM solution leads to less overlaps of loops and is easier to handle intuitively. On this example, additional moment is added as the Complete Graphic Statics method allows, this is accomplished with the addition of out-of-plane shear forces, or directly by adding M_0 to the loop. Total sum of forces and moments manifest equilibrium.

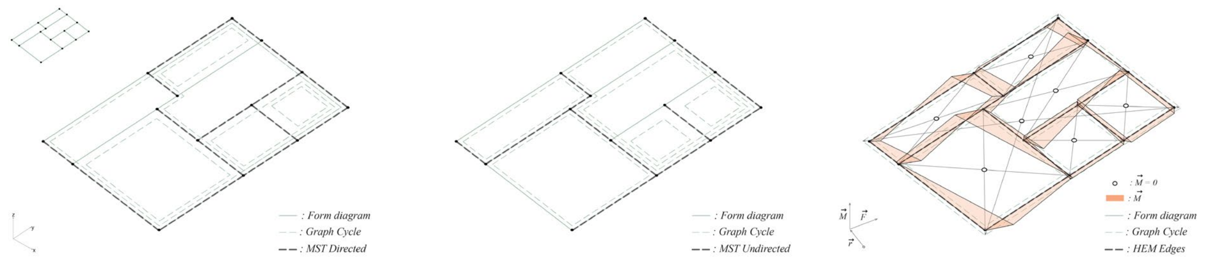

Fig. 7: Comparison of solutions for the case of a rectilinear topology, overlaps of Graph cycles show the clarity of the Half-Edge Mesh solution.

As discussed, eliminating time-consuming algorithms allows for more complex topologies. Figure 8 illustrates a randomly generated triangular mesh beyond the mere *xy* plane. The complete method diagram is created following the HEM construction. The clarity of the diagram manifests the simplicity of the application. Moment and forces can be designed and visualised in real time together with the modification of the initial topology. This can unlock potential in further advancing the method with heuristic algorithm applications to minimise or direct moments to specific locations and/or to develop moment/form-finding techniques, a novel introduction to existing Graphic Statics methods.

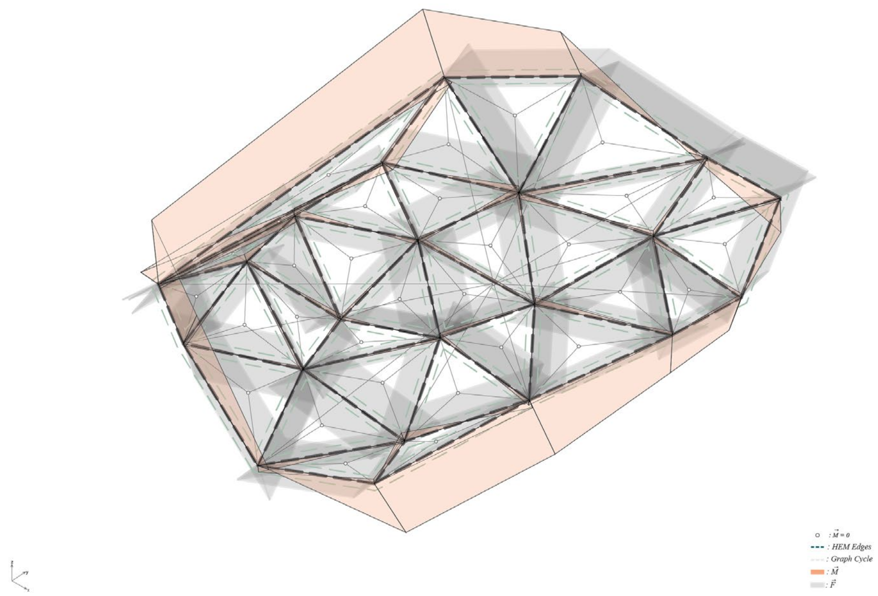

Fig. 8: The Half-Edge Mesh solution can be used for more complex topologies, a 3D example. The Graph cycles (loops) are translated into all axes the to construct force diagrams, lever-arms from each loop’s centroid are drawn to construct the moment vectors along the loop. Total force and moment along a member are the sum of its corresponding loops’ segment forces and moments.

4. Conclusion

Following the complete Graphic Statics method for frames and its Spanning Tree implementation, this research further investigated certain types of Spanning Trees to apply on simple 2D example topologies. This was sought through directed and undirected Minimum Spanning Trees. These were critically evaluated along the lines of facilitating the main Graphic Statics method for frames. It was argued that both solutions require costly algorithms that may eliminate the complexity of the introduced topology as well as lead to less-intuitive configurations for the Graphic Statics method. Among the two, the undirected Minimum Spanning Tree solution certainly is more intuitive to use due to its possibly least number of loop overlaps. The research also introduced the use of Half-Edge Mesh construction to advance the complete Graphic Statics method. It was argued that this novel introduction leads to more intuitive topology configurations and at the same time, due to its lower cost, it can allow for more complex topologies and real-time design applications. An example of a two-and-a-half dimensional canopy-like triangulated structure was shown. This investigation can be further explored with heuristic methods to steer a moment/form-finding method. Further investigation can be done to seek potential on 3D cases both regarding Spanning Trees as well as on 3D polygon mesh construction.

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