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Muqarnas 2.0: Topological Design of Stackable Polyhedral Blocks for Reconfigurable Masonry Corbell-Squinch-Vault Structures

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Abstract

Building one-off free-form or form-found vault shapes requires various custom non-reusable structural elements (or formworks) and an expert workforce. The aim of this paper is to propose an alternative direction which is about improving the constructability of the vaults and the versatility of the shapes that can be approximated rather than optimizing one-off free-form shapes for ideal vaults. This research introduces a topological design process for shaping muqarnas-like vault pieces that can be mass-produced for approximating suitable vault forms (a.k.a. funicular forms). The idea is to derive a small set of shapes with which one can combinatorically create a large plethora of approximated vaults, akin to muqarnas vaults in Middle-Eastern and North African architecture traditions. The gist of the proposed process is to start from vault or dome tessellations (a.k.a. tilings), inspired by Gothic and Persian vaulting patterns; mark the apex points and/or the top ridges of the vault; find load-transfer paths from the faces (tiles) incident to the top of the vault to the faces incident with the supporting walls on a dual graph corresponding to the base tiling through graph traversal methods; and then convert the pairs of consequent faces corresponding to the links in the topological load-paths to the base meshes for creating the shapes of the stackable corbel-vault pieces; and later carve out cornices (the Greek namesake of muqarnas vault pieces).

Keywords: Structural Topology, Structural Morphology, Reconfigurable Structures, Mass Customization, Funicular Structures, Combinatorial Design, Muqarnas Vault

1. Background

Masonry architecture has been in the spotlight of research for the past several decades due to its potential for reducing carbon emissions in the building industry [1]. However, the consideration of form-finding and one-off optimization of masonry forms is significantly higher than the little attention paid to the constructability and reconfigurability of masonry forms. The aim is to propose a topological design method for producing modular vaults with pieces that can be mass-produced as masonry blocks or stackable blocks to be made out of lighter materials. This change of focus from geometric perfection of vault shapes to the generative topological design [2] of vault segments is precedent in medieval architecture, especially in Gothic and Persian architectural schools, where a base tessellation forms the basis of the Gothic Rib Vault or the Persian Muqarnas Vault [3].

1.1. Evolution of vaults

Medieval architecture, particularly from the Gothic era, is esteemed for its unique structural qualities, with ornamentation fulfilling the structural functions. The most remarkable property of these structures is perhaps their constructability due to their topological segmentation that makes the stereotomy of the vault pieces approachable. As Clarence Ward explores, different vault types, including barrel, groin, rib, fan, and dome, played significant roles, each with distinct construction methods and architectural implications in achieving the ambitions of Gothic architects to create magnificent interiors with large span masonry vaults and large windows [4].

Romanesque architecture used domes, tunnel vaults, and groined vaulting. Gothic architecture introduced ribbed vaulting, allowing for complex designs such as six-part and four-part vaults. Transept vaulting differed from nave vaulting, often using half domes or tunnel vaults. Churches at intersections of naves and transepts featured crossing domes supported by pendentives (including spandrels) or squinches.

Variations in dome construction, such as ribbed domes and Gothic domes with double chevets, showcased the ingenuity and diversity of medieval vaulting techniques. These architectural innovations not only addressed structural challenges but also contributed to the aesthetic and spatial qualities of medieval churches and cathedrals.

The general idea of vaulting is to create a ceiling profile that can bear loads in such a way that the parts forming the vault are only (or mostly) in compression. Therefore, most vaults tend to resemble a form-found shape, although not exactly, mainly due to constructability issues. The concept of a modular vault, such as the muqarnas vault or a rib vault, can be very intriguing. The key point of the muqarnas vault is that with smaller modular pieces, one can, in principle, approximate many different types of vaults for various spans. Thus, it is possible to have a vault that is composed of modular pieces, is rather easily constructible, and, offers the flexibility to approximate a wide range of shapes. The approximation of good forms for vaults or funicular forms is achieved through attempts to closely mimic or meet structurally optimal shapes, using modular components.

1.2. Nomenclature

This chapter communicates the specific meaning of some essential terms used in this research and acts as a guide to the terminology employed.

1.2.1. Vault

Topologically, a vault is a concave surface structure that can cover a space, characterized by the ability to transfer the load of its own weight and possibly some structure above it as vertically as possible through the walls or piers supporting it to the ground. Vaulting techniques are arguably the oldest architectural inventions from the Mesopotamia region where wood has always been scarce as a building material. It covers a wide range of geometrical types, including domes. Unlike domes, vaults are not constricted to have a semi-sphere type of structure or have a circular projection. Vaults are structural or superficial structural architectural elements that usually determine a roof or a ceiling (or both). The different types of vaults are defined by the projection, topology and construction methods.

1.2.2. Spandrels

Spaces with a roughly triangular profile that are located between the vault (ceiling) and the floor that it supports atop of itself. Multi-storey masonry structures typically have filled-in spandrels that help with making the load forces more vertical in the piers and walls (see e.g. the Agha-Bozorg school in Kashan, Iran).

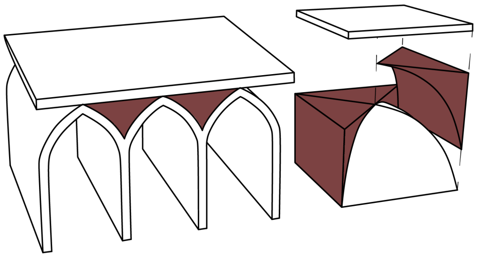

Figure 1: Spandrels (in brown) for different vault types

1.2.3. Apex

The apex is the highest point or vertex of the structure. In architecture, this term is used to describe the topmost point of the architectural feature: the peak where the sloping sides of the triangular roof meet, the highest point of a vault, or a dome. It holds significance from the topological point of view as it represents the point of convergence, connectivity, and singularity in a tessellation or tiling pattern that forms the basis of a rib vault or a muqarnas vault. It also plays a crucial role in determining structural stability, because it defines a point where the structure's load-bearing capacity is the most critical.

1.2.4. Tessellation

A tessellation or a tiling pattern describes the partitioning of a surface or a plane polygonal shape into smaller interconnected geometrical shapes, that are called tiles, cells, or faces, filling the whole space without any gaps or overlaps. A tessellation can be properly modelled as a polygon mesh in computer graphics. It can be achieved by repeating one tile, a set of different tiles, or using a mathematical mesh subdivision method, such as Voronoi, Catmull-Clark subdivision, quadrilateral or triangle subdivision patterns, and even arbitrary polygonal surface partitions as long as they can produce a valid polygon mesh.

1.2.5. Muqarnas

Muqarnas vaults or domes (or half domes) are typically intricate geometric designs prevalent in the Middle Eastern and North African architecture, that are formed by interlocking and overlapping smaller units - muqarnas cells. The term muqarnas is an arabesque word originated from the Greek word Cornice (a scooped out concave decorative pattern in the seams of walls and ceilings). They serve both a decorative function, due to their high degree of ornamentation, and a load-bearing structural function [5]. The structural function of muqarnas pieces is more clearly understood in the older examples of muqarnas in which the they are meant to segment a vault into smaller pieces to make it constructable, e.g. in the entrance Iwan of the Jame' Mosque in Isphahan, Iran. Muqarnas vaulting can be traced back to Persian and Armenian squinches for fitting circular domes into square shaped floor plans. Muqarnas vaulting technique is also influenced by the corbel dome method, seen in constructions such as the thumb of the Jewish prophet Daniel in Susa, Iran. Here, successive offset perimeters of polygonal room walls define stackable blocks forming quasi-conic roof shapes.

In short, the muqarnas vaulting technique can be considered as a combination of successive corbelling and squinch segmentation of vaults. This paper proposes a free interpretation of the idea of muqarnas vaulting with the intention to introduce a topological design method for producing muqarnas-like patterns for modular approximation of structural vaults. However, the aesthetic or decorative aspects of muqarnas vaulting remain out of the scope of this research.

2. Introduction

Implementation of modular systems and allowing for prefabrication of the discrete blocks aims to make the construction processes simpler and more accessible for non-experts and enable efficient on-site assembly. At the same time, configuring for stackable construction intends to create a possibility of reducing the number of used materials to a minimum. This promotes a more sustainable construction by decreasing the environmental footprint and waste production. Designing for reconfigurability also increases the reuse potential of the blocks.

It is particularly challenging to design reconfigurable voussoir geometries for vaults and their 3D spandrels to enable building on top of the vaults. Muqarnas blocks, even though typically used decoratively, are rooted in the stereotomy of squinches in Persian and Armenian architecture, respectively for adobe and stone constructions. Having evolved into geometrically intricate and aesthetically rich architectural elements, muqarnas originated from squinches, which served primarily as structural elements, facilitating the transition from rectilinear spaces to domes or vaults, or rectilinear spaces to curvilinear spaces [6]. The inherent modularity of muqarnas or corbel-squinch blocks allows for devising a system of reconfigurable blocks. It is precisely this potential that is utilized in the proposed methodology for designing sets of stackable blocks for vaulting.

Inspired by Bitting's research on dry-stackable blocks for dome construction [7], this study outlines a workflow for designing funicular masonry vault-like structures through topological configuration and optimization of polyhedral stackable blocks. This paper aims to consolidate and generalize the idea of muqarnas vaulting from being geometric and ornamental into a topological and structural concept.

3. Vaulting with Discrete Blocks

Vaulting with discrete blocks can either be aimed at approximating free-form or form-found vaults or be a process of combinatorial design for creating constructible and stable vault forms out of modular pieces. However, as the notion of form-finding is more widely known than combinatorial design of stable forms, these approaches are discussed from the perspective of approximating ideal free-form shapes or form-fund thrust networks. In computationally designing discrete modular blocks for approximating free-form vaults that are structurally valid, three distinct approaches can be identified. Determined by the chosen approach, the resulting vault will have a different construction method, block connections, and load transfer schemes.

3.1. 3D Space-filling Polyhedral Approach

The space-filling polyhedral approach is based on creating a 3D tessellation of space. This is possible by either copying and shifting the primary polyhedra (cubes, hexagonal prisms, elongated dodecahedrons, rhombic dodecahedrons, truncated octahedrons) [8] or by using a combination of space-filling non-primary polyhedra, such as {tetrahedra and octahedra}, {octahedra, truncated octahedrons, and cubes} [9]. The idea of polyhedral approximation of vaults is to create a 3D space filling or honeycomb grid and choose the cells that intersect with the form-found thrust network lines as the cellular block configuration that traces the thrust network or form-found surface adequately from a topological stance [10].

3.2. 3D voxel-based approach

Starting with the topological interlocking principles [11], the voxel-based approach begins with creating a 3D space-filling grid out of coloured/labelled chunks of voxels. However, unlike the 3D space-filling polyhedral approach, it focuses on regular discretization of the base grid cells for utmost modular approximation and refinement. Liu et al. explain the computational procedure and propose a framework, showcasing the resulting stackable discrete blocks for creating vault-like ceilings [12].

3.3. 2,5D muqarnas tessellation-based approach

The 2.5D muqarnas tessellation approach, also known as the LEGO®-like method, is described in detail in the work of Bitting, Azadi and Nourian [7], [10]. It starts with the 2D tessellation on a plane, using a chosen tiling pattern. The pattern is then navigated to the chosen highest point (apex). On this path, every cell of the tessellation is combined with the following coincident cell, representing the outline for the resulting 3D block. The overlapping (or corbelling) nature of the vault is needed to ensure the stackability of the blocks to create a 3D vault-like structure. To ensure structural validity and stability, the starting surface-mesh is dynamically relaxed (i.e. through Dynamic Relaxation, [13]) and the block positioning heights are approximated to follow the curvature of the resulting shell (Thrust Network).

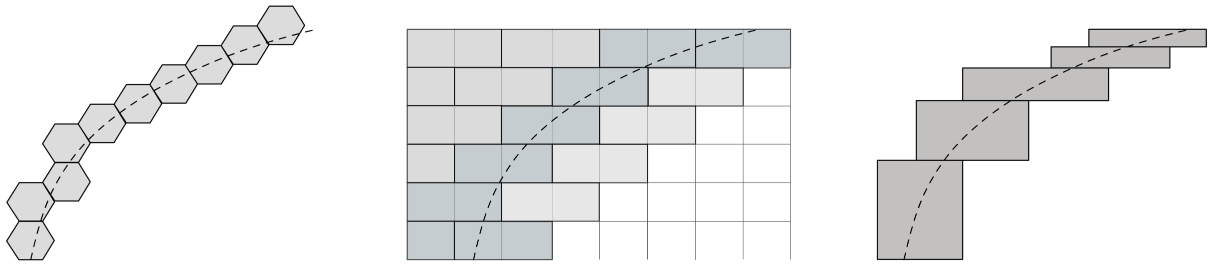
1. 3D Space-filling Polyhedral Approach. 2. 3D Voxel-based Approach. 3. Muqarnas Corbel Approach.

Figure 2: Vault approximation approaches (Thrust Network in dotted line)

4. Global Framework Methodology: Workflow for Creating Muqarnas-like Blocks

Building on Bitting's work [7], this research continues the exploration of the 2,5D approach. The broad methodological framework for designing the polyhedral blocks for funicular vault-like ceiling structures in the chosen context can be divided into three fundamental phases. The first phase focuses on the primary tessellation of the 2D plane, resulting in the base tiling. Phase two starts with the tessellated space obtained from phase 1 and aims to generate a set of reconfigurable blocks for stacking computationally. The third phase can be added for structural validation of the vault and designed blocks.

4.1. Phase 1: Tiling

The whole process of 3D muqarnas or squinch-corbel block generation depends on a topologically valid tiling of a planar polygon delineating the outline of the walls or piers that are to support the vault. Without loss of generality, the focus of this paper is restricted to cases in which the vault is solely supported by walls, notwithstanding the possibility that the walls may have arched openings inside them but not considering the cases where the openings in the walls directly coincide with the vault. A valid tiling or a tessellation pattern to be used for the squinch-corbelling or muqarnas vaulting must in principle be a valid polygonal mesh. However, for convenience, the attention is restricted to meshes only consisting of quadrilateral and triangular faces. From a different perspective, the importance of base tiling patterns can be seen in their effect on the ultimate structural performance of the form-found mesh (or thrust network) that is made up of the edges in the tiling pattern. As Oval has shown, the different tiling patterns can make a significant difference in the ultimate performance of shells and vaults, especially because of the initial placement of the apex vertices of the topological singularities [14].

4.1.1. Minimal Quadrilateral Tessellation

The notion of basing the initial tessellation pattern on a uniquely defined intrinsic tiling of the vault projected surface is quite appealing from a theoretical point of view because such innate quadrilateral tiling bases can be found from the topological skeletons of the base surface to be spanned with a vault, like the medial axis [15]. This idea of minimal quadrilateral tessellation on the basis of a medial axis comes from the work of David Rigby from NASA [16] and the procedure suggested by Oval et al. [17] for computing such quadrilateral tessellations from a base Delaunay tessellation of the base surface.

4.1.2. Carpenter's Tiling

Alternatively, the base tessellation can be started from another kind of topological skeleton such as the straight skeleton proposed by Aichholzer et al. [18] but the medial axis, which is also a topological skeleton, proposed by Blum [15] is more appealing from a mathematical stance due to its tight connection with the dual Delaunay and Voronoi tessellations and their relations to the distance fields on a surface. However, it is possible to think of the process of finding the topological skeleton of shapes as the same process that a carpenter would go through to design a hipped roof for the surface that is to be vaulted. That step can be called the carpenter's tessellation. This tessellation is effectively almost identical to what is achieved from a straight skeleton or essentially tracing the vertices of the base polygon when it is offset consequently in small steps inwards [19].

4.1.3. Repetitive Geometric Tiling Patterns

From an architectural engineering perspective, it might be more pragmatic to start the base tiling by repeating geometric polygon shapes in regular tiling patterns, e.g. as in the Penrose tiling pattern [20]. This approach is traditionally known as Girih-Chini in Persian Architecture [21], [22], [23]. The main advantage of this approach is that it is a priori known that the generated block shapes will only have a few known base shapes in the set of Girih tiles. In this way the resulting shapes will be utmost modular and thus the reconfigurability potential is theoretically raised. This approach is also followed by Bitting et. al. [7], by using a Penrose tiling [21].

4.2. Phase 2: Blocks

The workflow for the second phase of the framework gets the base tiling as input from the first phase and consists of several distinct steps:

4.2.1. Mesh Quadrilateralization

Depending on the tessellation obtained from phase 1, the starting mesh may need to be further subdivided into quadrilaterals for various reasons. The higher the level of subdivision, the smaller the final blocks will be, therefore resulting in the higher number of pieces for vault approximation. Quadrilateral mesh subdivision is preferable because it is more convenient, especially because of the dualization and the intended staggering or brick-laying overlap of the muqarnas blocks. So, the alternating layers can have a foot under the shoulders of the layers below them and create a pattern of stackable and possibly even interlocking blocks that can be dry assembled.

4.2.2. Finding a Dual Graph

From the base tessellation, the dual graph is generated, representing the connectivity and neighbouring or adjacency relationship between the faces of the base mesh (edge-sharing relations). The dual graph also serves as the guide for further creating the blocks that go in-between the layers (in the z direction) of the blocks generated from the base mesh. The duality ensures stackability for creating a valid approximated corbelled vault-like ceiling.

4.2.3. Directed Graph Traversal for Load-Path Finding

A crucial step in the process is to perform graph traversal from the nodes which are dual to the faces that are incident to apex vertices of ridge edges (origins) to the faces that reach the walls (destinations). This ensures finding the optimal and easiest load-paths for transferring the gravity loads resulted from the self-weight and the dead-weight put atop of the muqarnas blocks as straight and quickly as possible to the faces that are supported by the walls, piers, or columns.

Without loss of generality, in the simple case of meshes that can be segmented into topologically concentric strips of faces surrounding the ridge lines or the apex points, there will not be any ambiguity in finding these strips because there will be a regular succession of descending tiers of blocks corbelled on top of each other. In this case, by using the Breadth-First-Search graph traversal, such topological layers of faces or strips of faces are found from the top to the bottom of the vaulting tessellation. In more sophisticated cases, the same idea of graph traversal will be leading, albeit leaving some room for arbitrary topological design decisions for picking the number of descending and locally ascending block levels, i.e. a technique sometimes used in muqarnas vaulting for creating impressive stalactite-like features.

4.2.4. Creating the 3D Blocks

Once a bunch of spanning directed load-paths are chosen in such a way that they would span through all the faces in the tessellation, possibly with some overlapping nodes in the paths (which would indicate the multiplication of the required compressive strength of the blocks dual to these nodes), then the paths can be decomposed into chain-links of pairs of consecutive mesh faces (tiles). Such pairs of consecutive tiles can then be used as the mesh base surfaces to be extruded to shape the basic prismatic forms of the corbel blocks. The actual height of these prismatic blocks actually depends on the form-finding or form-approximation process which is a straightforward geometric process that falls out of the scope of this part of the process. For now, it is assumed for the blocks to be arbitrarily tall enough to allow for geometric adjustment later on.

4.2.5. Cornicing

The term muqarnas is the Arabized version of the Greek word Corniced. The structural muqarnas blocks are in-fact nothing but the corniced corbel vault blocks. The actual scooping of excess mass in the corbel vault pieces that is not structurally effective in the gravitational load-transfer is here referred to as scooping. Ahu-Paay, the Persian term for muqarnas, means "dear-legged," referring to its resemblance to a deer's arched legs used in carving corbel blocks. Thus, this step is referred to as Cornicing which transforms the prismatic blocks of the corbel vault found in the previous step into corbell-squinches or muqarnas blocks in this step. However, instead of such geometric approaches for scooping excessive masonry mass out of the corbel vault pieces, this task is approached as a matter of removing a cellular chunk of excessive volume from the mesh blocks and then smoothing the remaining mesh block through mesh relaxation or mesh subdivision algorithms.

4.3. Phase 3: Form-Finding

Notwithstanding the possibility of freely starting from the set of created blocks to combinatorically create a modular vault in a valid (though not necessarily optimal) shape, we instead focus on approximating idealized, form-found thrust networks that could be considered topologically dual to the interconnected and stacked levels of a muqarnas block. As mentioned in the previous phase, muqarnas blocks begin with undetermined prismatic heights. Thus, the form-finding process aims to establish these heights for the prismatic corbel blocks. This process involves forming a hypothetical thrust network consisting of lines that trace the edges of the base tiling mesh used to create the prismatic shapes of the muqarnas blocks. We relax or optimize the shape of this network to counteract upended, gravity-like forces by applying spring-like tensile strengths to the line segments, proportional to the force densities and stiffnesses of the compressive links between the blocks (idealizing the mortar or dry joints between the blocks and the dual pairs of half-cells from the blocks). The network's shape is then topologically approximated by intersecting it with prismatic polyhedral shapes extruded sufficiently from the base meshes drawn for the muqarnas blocks. This intersection determines the midpoint height as the centre height of the muqarnas block and establishes the actual extrusion height based on its tier step and this height value.

For the actual form-finding process on the thrust-line network various approaches can be employed, such as the non-iterative Force-Density Method by Scheck [24] or the iterative Dynamic Relaxation Method by Barnes [25]. However, realistically, our proposed form-finding process requires an estimate of the volume and, consequently, the weight of the blocks (considering a material choice with a known density) to proceed effectively. While it may not be ideal for precise shape optimization, the concept of approximate vaulting remains intact. By designing the muqarnas blocks and the vault to be taller than necessary, we can better manage vertical load forces, ensuring a valid form, if not an optimal one.

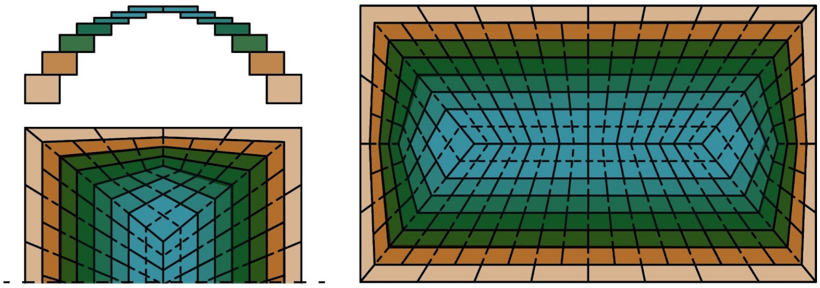

Figure 3: Stacked blocks created from a tessellation (concept)

5. Designing Stackable Reconfigurable Blocks

The methodology proposed in the previous section is in fact quite general and can be applied on any kind of valid tessellation (be it a simplicial complex or a valid manifold polygon mesh). However, since our goal here is to provide for utmost modularity, constructability and the possibility of creative combinatorial design, we are restricting our case study to a highly modular and regular base shape. Because we anticipate this restriction to help reduce the number of different types of resulting blocks. For the sake of brevity, the scope of the case study is restricted to phase 2 of the global workflow. A detailed procedural workflow is implemented for this phase, showcasing a proof of concept with the final stackable polyhedral blocks designed as meshes.

5.1. Base Tiling

Here, an illustrative example for the creation of muqarnas-like blocks on the basis of a Carpenter's tiling pattern is demonstrated. The primary carpenter's roof is drawn manually, where one face corresponds to one planar surface of the roof. The starting mesh is then drawn, subdividing some faces of the roof further to satisfy the mesh validity requirements of vertex, edge, and face connectivity. The base mesh is then made up of this base tiling by the means of quadrilateralization process of mesh subdivision for two iteration steps.

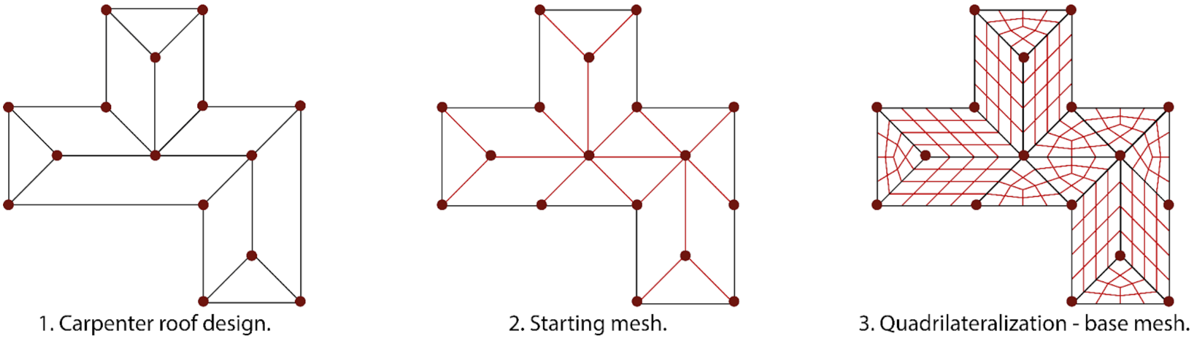

Figure 4: Generation of the base mesh

5.2. 2D Block Creation

This section presents the steps followed to create the 2D base mesh of the prismatic blocks in a few smaller steps. Firstly, the concentric strips of faces from the faces incident to the ridge lines to the strip of mesh faces adjacent to the walls that are supposed to support the vault. These strips are implicitly based on the load paths from the ridges towards the walls. However, for creating a stable vault with staggering brick-bonds, the duo bundles of mesh faces are made to create the bottom bases for the prismatic muqarnas blocks out of the base mesh and its dual in alternating step levels. Then, overlapping and shifted layers of such quadruple bundles of faces are created from the walls towards the top of the vault until we reach a full coverage.

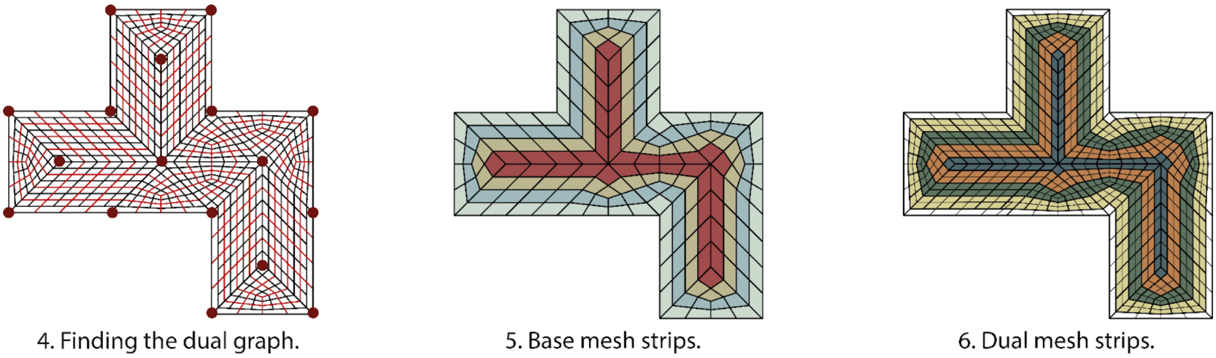

Figure 5: Sorting of base and dual mesh strips

5.3. 3D Block Creation

In this step focuses on extruding the prismatic base blocks for the muqarnas pieces tall enough in such a way as to ensure that their centres would be higher than the intersection points achieved with the very tall prisms on their bases with the form-found thrust line network. Afterwards, it is possible to systematically scoop out pieces of excess mass from the bottom side falling below the intrados of the vault and smoothen the shapes of the pieces for manufacturing and aesthetic purposes.

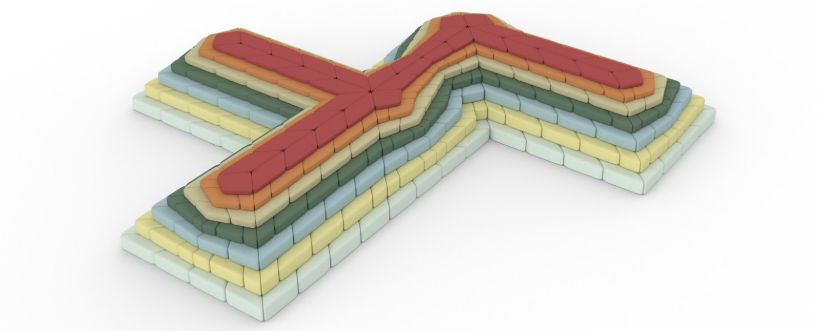

Figure 6: Resulting stacked blocks for roof approximation

6. Discussion and Future Work

The proposed methodology merges historical vaulting techniques with topological principles to design modular vaults efficiently, leveraging generative design for mass production of masonry structures, addressing construction challenges, and aligning with sustainability goals to reduce carbon emissions and material waste. While demonstrating the feasibility of this topological design approach and outlining the methodology, this paper does not claim to replicate historical muqarnas vaults. Instead, it seeks to draw inspiration from Persian muqarnas and Gothic ribbed vaults to develop a new modular vaulting technique that supports mass production and customization of vault shapes. This designer-driven process incorporates considerations of constructability and aesthetics, emphasizing a topological design with mesh topology and interlocking brick-bond patterns as key elements.

Future research could explore ways to generalize or stylize the process, focusing on the derivation of base tiling patterns from the intrinsic properties of base surfaces and the bundling of mesh tiles to create overlapping, space-filling polyhedral blocks, especially for filling spandrels. Additionally, automating structural validation of muqarnas vaults for the combinatorial design using mass-produced blocks warrants further investigation.

The authors admit to be facing technical implementation challenges for making the cornices in the pieces due to the limitations of Rhino7 of accepting mesh faces with a higher edge number than 4. This issue is planned to be solved for the camera-ready submission with the use of Python.

Acknowledgements

For the implementation of the experimental workflow, several tools were used. Firstly, the starting base mesh (tessellation) was drawn manually in Rhino3D. Further steps of the process were completed in Grasshopper3D with the help of some Python code snippets and several open-source plugins: Weaverbird, Ivy, Stripper, and Mesh+. Herewith we thank the developers of these plugins for the useful methods provided in them.

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