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# Legendre transforms for graphic statics with moments

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**Abstract**

In his classic papers of 1864 and 1870, Maxwell describes the mathematics that underlies the continuum version of graphic statics as a mapping between a body space and a stress space. Although Maxwell does not use the phrase, this mapping is a Legendre transform. It is defined via the gradients and intercepts of a pair of dual functions. These are dual Airy stress functions for 2D stresses and dual Maxwell-Rankine stress functions for 3D stresses. In the discrete limit (i.e. piecewise linear but continuous stress functions) these become the polyhedral Airy and the polytopic Rankine stress functions for describing axial forces in 2D and 3D trusses. The dualities are those inherent to 3D and 4D projective geometry respectively. As is well-known, bar forces are given by reciprocal line lengths for 2D trusses and by areas of reciprocal polygons for 3D trusses. Less well-known is the fact that these stress functions also encode information about moments. This is illustrated for some fundamental examples in 2D and 3D.

**Keywords**: Legendre transform; graphic statics; bending moments; Airy stress function; polytopic stress function.

## 1. Introduction

The familiar Form and Force diagrams of graphic statics are Legendre transforms of each other. In his 1864 and 1870 papers[1, 2], Maxwell generalised graphic statics by looking at continuous stresses in 2D and 3D solids. He then showed how the piecewise-linear limit of this description, with curvature concentrated at edges and nodes, leads to the graphic statics of trusses, with axial forces given by reciprocal line lengths in 2D or reciprocal polygonal areas in 3D. Maxwell put great emphasis on duality. He demonstrates that 2D reciprocal diagrams are projections of 3D plane-faced polyhedra - the polyhedral Airy stress function and its dual. Maxwell noted that these dual polyhedra are related via a polarity with respect to a paraboloid of revolution, a concept from projective geometry. Maxwell did not mention Legendre. (At around that time, Chasles [3] noted the equivalence between the polarities and Legendre transforms. The relevance of Legendre for graphic statics was brought to our attention by [4]). Maxwell also did not show that his theory contains a geometric description of moments as well as forces.

At previous IASS Symposia [5, 6] the first author presented complete theories of graphic statics and kinematics based on the six projections of loops in 4D space. Those descriptions used homology theory and exterior calculus, with Virtual Work manifesting itself as a "top form" obtained from the wedge product of force and displacement loops. The present paper shows that this loop description is largely contained within Maxwell's original papers, with the duality embodied by Legendre transforms.

## 2. What is a Legendre transform?

In 1D, the Legendre transform of a function $F(x)$ is the function $\phi(\xi)$ defined by $\phi = x\xi - F$ with $\xi = dF/dx$. The point $x, F$ maps to the point $\xi, \phi$. This is illustrated in Fig. 1. The new coordinate $\xi$ is

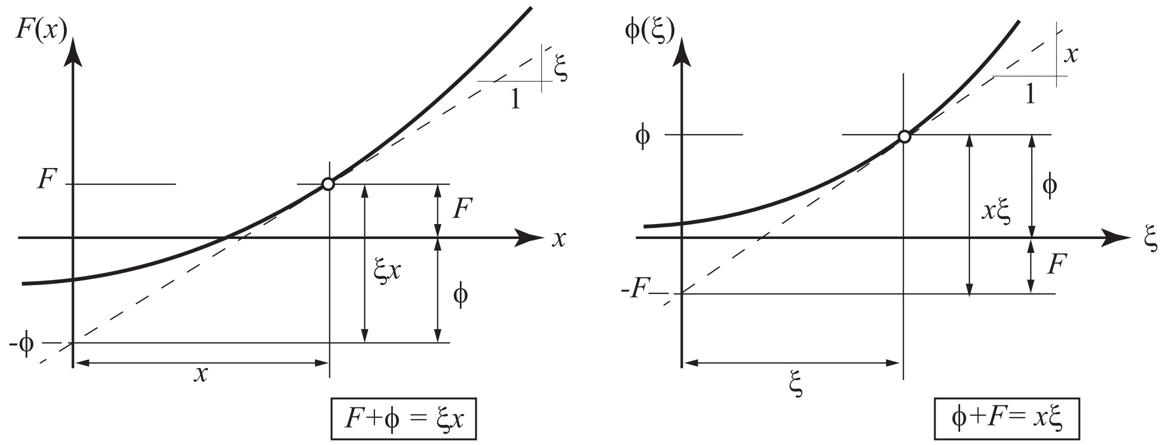



Figure 1: The function $F(x)$ can be used to define a dual function $\phi(\xi)$ where $\xi$ is the slope of $F$ at $x$, and $\phi$ is the (negative of the) intercept of the tangent to $F$ at $x$ with the $F$ axis. The functions $F(x)$ and $\phi(\xi)$ are Legendre transforms of each other.

given by the slope of the function $F$ at $x$, and the value of the function $\phi$ there is equal to (the negative of) the intercept of the line tangent to $F$ at $x$ with the $F$ axis. That is, rather than defining the curve in terms of pairs $(x, F)$ we may define it via the pairs (gradient, -intercept) which, when plotted, define a dual curve $(\xi, \phi)$.

The construction generalises readily to higher dimensions. Given a function $F(\boldsymbol{x})$ where $\boldsymbol{x} = (x, y, z, \ldots)$, the Legendre transform is $\phi = \boldsymbol{x}.\boldsymbol{\xi} - F$ with $\boldsymbol{\xi} = \mathrm{grad}\ F$. The dual coordinates $\boldsymbol{\xi} = (\xi, \eta, \zeta, \ldots)$ are again given by the slope and the value of the dual function $\phi$ is again given by (the negative of) the intercept of the plane tangent to $F$ at $\boldsymbol{x}$ with the $F$ axis. The dual coordinate systems have the same spatial orientation ($\xi$ is parallel to $x$, etc.). Thus, the original function may be defined as $(\boldsymbol{x}, F)$ and the dual function $(\boldsymbol{\xi}, \phi)$ is again the (gradient, -intercept).

This somewhat abstract transformation finds much use in physics, transforming between Lagrangian and Hamiltonian formulations of mechanics, say, or between the various potentials of internal energy, enthalpy, Gibbs and Helmholtz free energies. In this paper, the Legendre transform will embody the duality between form and force.

## 3. The body space and the stress space

Stress functions are familiar objects in the theories of 2D and 3D solid mechanics: typically there is some function $F$ whose various second derivatives define the stresses within the solid. In his 1870 paper, Maxwell[2] invokes the notion of such a stress function, but focuses instead on its first derivatives.

He begins with physical 3D space - the "body space" - within which the stresses are to be defined, and defines a mapping from this body space to an imagined space he calls the "stress space". The mapping is defined by a function $F(x, y, z)$, the stress function, over the Euclidean body space with coordinate axes $x$,$y$,$z$. That is, for any differentiable function $F$, there is a dual function $\phi(\xi, \eta, \zeta)$ defined by

$$\phi = x\xi + y\eta + z\zeta - F \quad \text{where} \quad \xi = \frac{\partial F}{\partial x}, \quad \eta = \frac{\partial F}{\partial y}, \quad \zeta = \frac{\partial F}{\partial z}\ . \tag{1}$$

This is a Legendre transform. This may be written more symmetrically as

$$F + \phi = \boldsymbol{x}.\boldsymbol{\xi} \quad \text{with} \quad \boldsymbol{\xi} = \mathrm{grad}\, F \quad \text{and} \quad \boldsymbol{x} = \mathrm{grad}\, \phi\ . \tag{2}$$

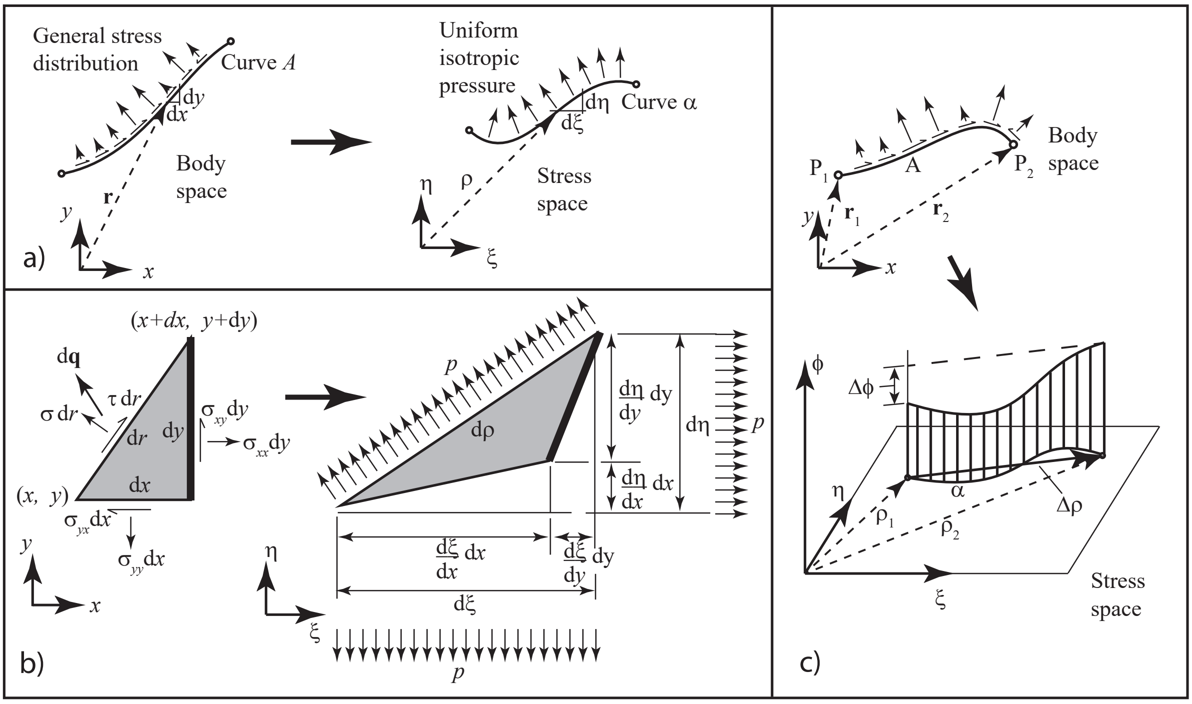



Figure 2: a) Curve $A$ in the body maps to curve $\alpha$ in stress space. b) Detail showing the forces on corresponding elements of each curve. Note how the thick line $dy$ maps to its sloping image. c) The vector $\Delta\rho = (\Delta\xi, \Delta\eta)$ between the end points of curve $\alpha$ gives (once scaled by $p$) the resultant force on $A$ rotated by 90°. The change $\Delta\phi$ in the dual stress function between the end points of $\alpha$ gives the moment of the force about the origin.

Maxwell describes how any surface $A$ within the body space is mapped to a surface $\alpha$ in the stress space. The mapping is such that the total force acting on $A$ due to the stresses within the body can - quite remarkably - be replicated by a field of isotropic pressure $p$ in the stress space acting normal to the image surface $\alpha$. In the body space, the stress field can consist of any equilibrium set of normal and shear stresses, but in the stress space, the only stress is the simple isotropic pressure. Thus, this remarkable stress function is purely geometric.

Fig. 2 shows the 2D case. The edge $dy\ \boldsymbol{j}$ carries a normal stress $\sigma_{xx}$ thus the $x$ component of the force on that edge is $\sigma_{xx} dy\ \boldsymbol{i}$. The image of that edge is

$$\frac{\partial \xi}{\partial y} dy\ \boldsymbol{i} + \frac{\partial \eta}{\partial y} dy\ \boldsymbol{j} \tag{3}$$

The projected length of that image that faces the $x$ direction is $(\partial\eta/\partial y)dy$, thus $\sigma_{xx} dy = p(\partial\eta/\partial y)dy$. Since the Legendre transform defines reciprocal coordinates to be the gradient of the stress function, (i.e. $\eta = \partial F/\partial y$) it follows that $\sigma_{xx} = p(\partial^2 F/\partial y^2)$. The other components may be determined similarly. That is, the familiar definitions of stresses via second derivatives of the Airy stress function follow directly from the Legendre transform between between body and stress space: Airy follows from Legendre. And what might be less familiar is the dual stress function $\phi$, which will play a significant role in what follows.

## 4. Moments in 2D continua

The state of stress defined by the Legendre transform naturally satisfies force equilibrium. Further, Maxwell[2] shows that moment equilibrium is also satisfied. Here we show how the moment information is contained simply within the dual stress function $\phi$.

Figure 2 illustrates the 2D case with stresses acting on one side of some curve $A$ within the body. Let $\boldsymbol{dq}$ be the total force acting across the element $\boldsymbol{dr} = dx\ \boldsymbol{i} + dy\ \boldsymbol{j}$ based at the location $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}$ as shown. It can be seen from the figure that $\boldsymbol{dr}$ maps to $\boldsymbol{d\rho} = d\xi\ \boldsymbol{i} + d\eta\ \boldsymbol{j}$. The force $\boldsymbol{dq}$ on $\boldsymbol{dr}$ is thus equal and opposite to $pd\eta\ \boldsymbol{i} - pd\xi\ \boldsymbol{j}$. The moment about the origin is thus

$$\boldsymbol{dm} = \boldsymbol{r} \times \boldsymbol{dq} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x & y & 0 \\ -pd\eta & pd\xi & 0 \end{vmatrix} = p(xd\xi + yd\eta)\boldsymbol{k}$$

Now, since form and force are a Legendre transform pair, $x = \partial\phi/\partial\xi$ and $y = \partial\phi/\partial\eta$ thus

$$\boldsymbol{dm} = p\left(\frac{\partial\phi}{\partial\xi}d\xi + \frac{\partial\phi}{\partial\eta}d\eta\right)\boldsymbol{k} = p\ d\phi\ \boldsymbol{k}\ .$$

The total moment about the origin exerted by the stresses on curve $A$ is thus the integral

$$\boldsymbol{M} = \int_{P_1}^{P_2} \boldsymbol{dm} = p\int_{P_1}^{P_2} d\phi\boldsymbol{k} = p\left(\phi(P_2) - \phi(P_1)\right)\boldsymbol{k}\ .$$

That is, the stresses on one side of curve $A$ create a moment about the origin equal to the difference of the dual stress function $\phi$ between its end points, scaled by the constant pressure $p$ in the stress space.

### 4.1. Example: Airy's beam

In Maxwell 1870 [2], Maxwell revisited a beam example from Airy's 1863 paper [7] which is referred to here as the Airy beam. Consider a simplified version of Airy's beam, with span $L = 2a$ and depth $b$, carrying a uniformly-distributed load $w$ per unit length (see Figure 3). It is supported by shear stresses on the end faces. A simple stress function that defines an equilibrium stress field satisfying these conditions is

$$F(x, y) = w\frac{(a^2 - x^2)}{2b^3}(3by^2 - 2y^3)$$

with the stresses given by the second derivatives $\sigma_{xx} = F_{,yy}$, $\sigma_{yy} = F_{,xx}$ and $\sigma_{xy} = -F_{,xy}$ thus

$$\sigma_{xx} = \frac{3w(a^2 - x^2)}{b^3}(b - 2y), \quad \sigma_{yy} = -\frac{w}{b^3}(3by^2 - 2y^3) \quad \text{and} \quad \sigma_{xy} = \frac{6wx}{b^3}y(b - y).$$

These match elementary beam theory: longitudinal stresses vary linearly with depth and quadratically along the beam length; vertical stresses fall from $w$ on the upper face to zero at the soffit; and shear stresses have a parabolic profile through the beam depth. This stress function does not satisfy the biharmonic equation $\nabla^4 F = 0$ that would be appropriate for a linear elastic system with no body forces [8]. However, we are considering equilibrium only and the chosen stress function provides a simple - and valid - system of equilibrium stresses.

As usual, second derivatives of $F$ lead to the stresses, from which principal stress trajectories can be computed (Fig. 3c). Less traditionally, the first derivatives of $F$ lead to Maxwell's Diagram of Stress in the stress space (Fig. 3d) as per Maxwell 1870 [2]. More unusual yet, the dual stress function can

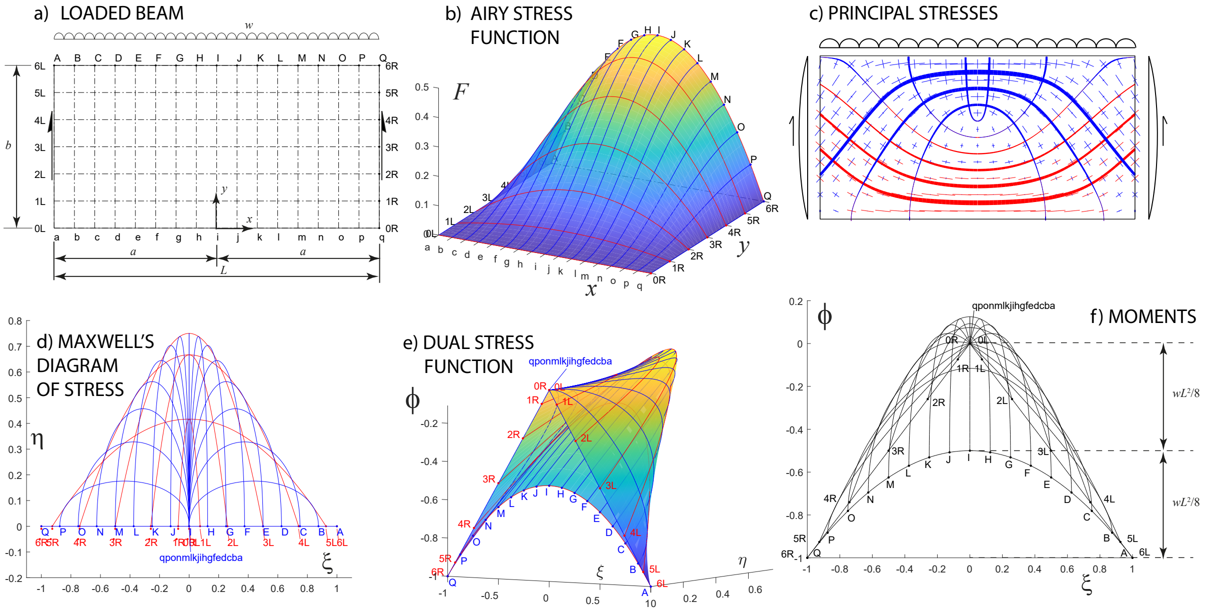



Figure 3: a,b,c) A loaded beam, an Airy stress function and the resulting principal stresses; d,e) Maxwell's Diagram of Stress and the dual stress function above it; f) Reading moments from the dual stress function.

be computed and plotted over the Diagram of Stress (Fig. 3e), allowing moment information to be read directly. For example, the midspan section $iI$ maps to the upper centre of Fig. 3f. There is no net force on this section, but the bending moment of $wL^2/8$ can be read from the dual stress function. Similarly the upper left surface $AI$ has no bending moment, but its force $wL/2$ has lever arm $L/4$ about the origin and, again, the resulting moment $wL^2/8$ can be read from Fig. 3f.

Fig. 4 illustrates how closely the continuum description matches the more familiar discrete case. The form and force diagrams for a tied-arch bridge are shown in Fig. 4a,c). The force diagram is Maxwell's Diagram of Stress. Fig. 4d shows the dual stress function, which is not usually computed in graphic statics. It is a discrete version of the continuum case of Fig. 3e. Again, moment information may be read from the dual stress function values, as was done in Fig. 3f.

## 5. Moments in 3D continua

Just as the preceding 2D stress analysis used surfaces in 3D, so a natural setting for 3D stress analysis is the 4D space with coordinates $(F, x, y, z)$. Consider two small vectors $\boldsymbol{du}_4$ and $\boldsymbol{dv}_4$ based at the point $(F, x, y, z)$ (see Figure 5)

$$\boldsymbol{du}_4 = dF_u\boldsymbol{h} + dx_u\boldsymbol{i} + dy_u\boldsymbol{j} + dz_u\boldsymbol{k} \quad \text{and} \quad \boldsymbol{dv}_4 = dF_v\boldsymbol{h} + dx_v\boldsymbol{i} + dy_v\boldsymbol{j} + dz_v\boldsymbol{k}$$

The *wedge product* of these two vectors defines the elemental area $\boldsymbol{dA}_4$.

We now map this from the 4D body space $(F, x, y, z)$ to the 4D stress space $(\phi, \xi, \eta, \zeta)$, with the images of $\boldsymbol{du}_4$, $\boldsymbol{dv}_4$ and $\boldsymbol{dA}_4$ being $\boldsymbol{d\psi}_{u4}$, $\boldsymbol{d\psi}_{v4}$ and $\boldsymbol{d\alpha}_4$ respectively. The image vectors are

$$\boldsymbol{d\psi}_{u4} = d\phi_u\boldsymbol{h} + d\xi_u\boldsymbol{i} + d\eta_u\boldsymbol{j} + d\zeta_u\boldsymbol{k} \quad \text{and} \quad \boldsymbol{d\psi}_{v4} = d\phi_v\boldsymbol{h} + d\xi_v\boldsymbol{i} + d\eta_v\boldsymbol{j} + d\zeta_v\boldsymbol{k}$$

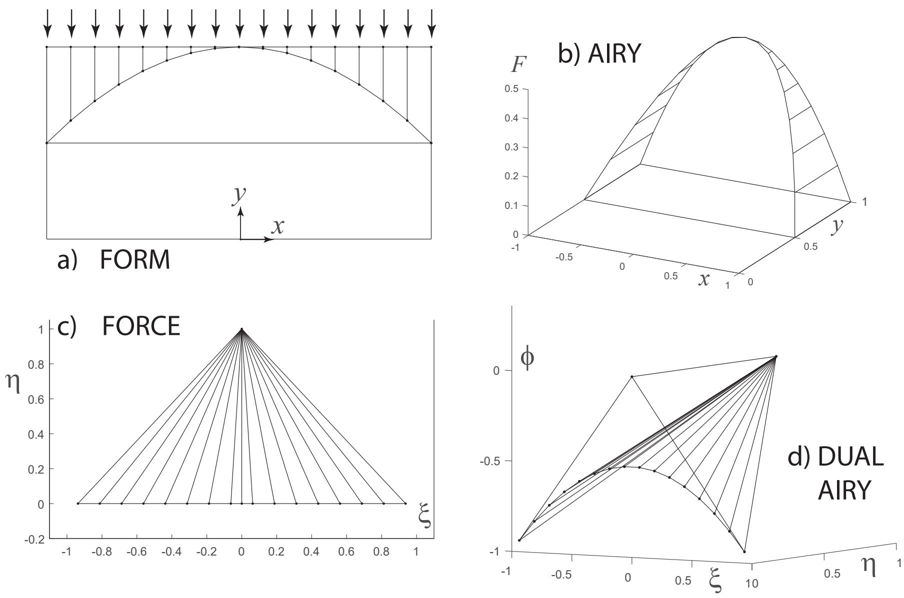



Figure 4: a) The Form diagram for a loaded, tied-arch bridge; b) a piecewise-linear Airy stress function; c) Maxwell's Diagram of Stress is the Force Diagram; d) The dual stress function containing moment information.

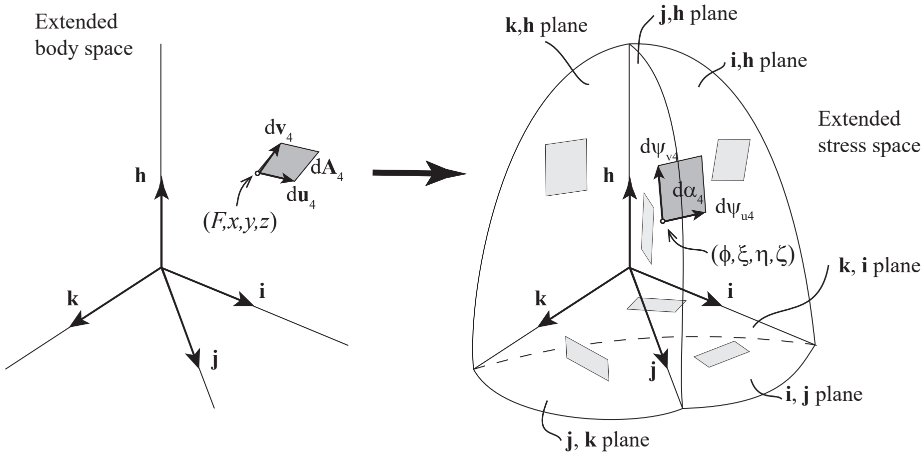



Figure 5: A 2D element $\boldsymbol{dA}_4$ being mapped to its image $\boldsymbol{d\alpha}_4$ in the 4D setting. The six independent projections of $\boldsymbol{d\alpha}_4$ are shown schematically.

where the first component of $\boldsymbol{d\psi}_{u4}$ is given by

$$\begin{aligned} d\phi_u &= \phi(\xi + d\xi_u, \eta + d\eta_u, \zeta + d\zeta_u) - \phi(\xi, \eta, \zeta) \\ &= [(x + dx_u)(\xi + d\xi_u) + (y + dy_u)(\eta + d\eta_u) + (z + dz_u)(\zeta + d\zeta_u) - (F + dF_u)] \\ &\qquad\qquad - [x\xi + y\eta + z\zeta - F] \\ &= (\xi dx_u + x d\xi_u) + (\eta dy_u + y d\eta_u) + (\zeta dz_u + z d\zeta_u) - \left(\frac{\partial F}{\partial x}dx_u + \frac{\partial F}{\partial y}dy_u + \frac{\partial F}{\partial z}dz_u\right) \\ &= (\xi dx_u + x d\xi_u) + (\eta dy_u + y d\eta_u) + (\zeta dz_u + z d\zeta_u) \ - [\xi dx_u + \eta dy_u + \zeta dz_u] \\ &= x d\xi_u + y d\eta_u + z d\zeta_u \end{aligned}$$

and the second component is given by

$$d\xi_u \quad = \quad \frac{\partial \xi}{\partial F}dF_u + \frac{\partial \xi}{\partial x}dx_u + \frac{\partial \xi}{\partial y}dy_u + \frac{\partial \xi}{\partial z}dz_u \quad = \quad \frac{\partial \xi}{\partial x}dx_u + \frac{\partial \xi}{\partial y}dy_u + \frac{\partial \xi}{\partial z}dz_u$$

because, in the first term, $\partial\xi/\partial F = (\partial/\partial F)(\partial F/\partial x) = 0$. The third and fourth components of $\boldsymbol{d\psi}_{u4}$ are similar, but with $\xi$ replaced by $\eta$ and $\zeta$ respectively. The components of $\boldsymbol{d\psi}_{v4}$ are similar, with $u$ replaced by $v$.

The oriented area of the image element $\boldsymbol{d\alpha}_4$ is given by

$$\begin{aligned} \boldsymbol{d\alpha}_4 \;=\; \boldsymbol{d\psi}_{u4} \wedge \boldsymbol{d\psi}_{v4} \;&=\; (d\phi_u\boldsymbol{h} + d\xi_u\boldsymbol{i} + d\eta_u\boldsymbol{j} + d\zeta_u\boldsymbol{k}) \wedge (d\phi_v\boldsymbol{h} + d\xi_v\boldsymbol{i} + d\eta_v\boldsymbol{j} + d\zeta_v\boldsymbol{k}) \\ =\; & (d\eta_u d\zeta_v - d\eta_v d\zeta_u)\boldsymbol{j} \wedge \boldsymbol{k} \;+\; (d\zeta_u d\xi_v - d\zeta_v d\xi_u)\boldsymbol{k} \wedge \boldsymbol{i} \;+\; (d\xi_u d\eta_v - d\xi_v d\eta_u)\boldsymbol{i} \wedge \boldsymbol{j} \\ & + (d\phi_v d\xi_u - d\phi_u d\xi_v)\boldsymbol{i} \wedge \boldsymbol{h} \;+ (d\phi_v d\eta_u - d\phi_u d\eta_v)\boldsymbol{j} \wedge \boldsymbol{h} \;+\; (d\phi_v d\zeta_u - d\phi_u d\zeta_v)\boldsymbol{k} \wedge \boldsymbol{h} \end{aligned}$$

The six components of the oriented area are now clearly visible. Three are on the basis planes of the usual 3D stress space spanned by $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$. The components of the force acting on $\boldsymbol{dA}$ in the $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ directions are given by $p$ times the oriented areas on the $\boldsymbol{j} \wedge \boldsymbol{k}$, $\boldsymbol{k} \wedge \boldsymbol{i}$ and $\boldsymbol{i} \wedge \boldsymbol{j}$ planes respectively.

The components of the moment exerted about the origin by the force on $\boldsymbol{dA}$ is given by the oriented areas on the other three basis planes. Specifically, the moment about the origin has components about the $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ axes given by $p$ times the components of oriented area on the $\boldsymbol{i} \wedge \boldsymbol{h}$, $\boldsymbol{j} \wedge \boldsymbol{h}$ and $\boldsymbol{k} \wedge \boldsymbol{h}$ planes respectively.

These results extend readily to finite surface patches to give the following simple yet deep result: for a 2D surface patch within the 4D extended body space of $(F, x, y, z)$, not only is the total force given by $p$ times the oriented area of its image within the usual 3D $(\xi, \eta, \zeta)$ stress space, but also $p$ times the oriented areas of its image when projected onto the $(\xi, \phi)$, $(\eta, \phi)$ and $(\zeta, \phi)$ planes give the components of the total moment about the origin exerted by the forces on that patch.

The method also extends naturally to the discrete case of frames carrying axial and shear forces, together with bending and torsional moments, as described in [9, 5]. Any frame can be readily decomposed into a set of loops, with surfaces spanning loops and cells bounded by surfaces. The stress function is piecewise-linear over each cells, and may be discontinuous across the surfaces between cells. Lines of action of any applied forces may be considered to be bars of the frame, these being connected to some remote rigid armature. Bars need not be straight and surfaces between cells need not be plane.

### 5.1. Example: a tetrahedral frame

The simplest example of a polytopic stress function is the regular tetrahedral truss with central node (Fig. 6a). It has the K5 graph. (This is the 3D version of Maxwell 1864 Fig. 1 which had a triangle with spokes, that being a K4 graph). We choose a continuous piecewise-linear stress function with $F = 0$ at the four outer nodes and $F = H$ at the central node. The length of an outer bar is $2L$.

The nodal coordinates of the dual stress function can be found by solving the Legendre equation 1 evaluated at any four nodes of a structural cell (see Konstantatou *et al.*, [10] )

$$\begin{bmatrix} x_1 & y_1 & z_1 & -1 \\ x_2 & y_2 & z_2 & -1 \\ x_3 & y_3 & z_3 & -1 \\ x_4 & y_4 & z_4 & -1 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \\ \phi \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}$$

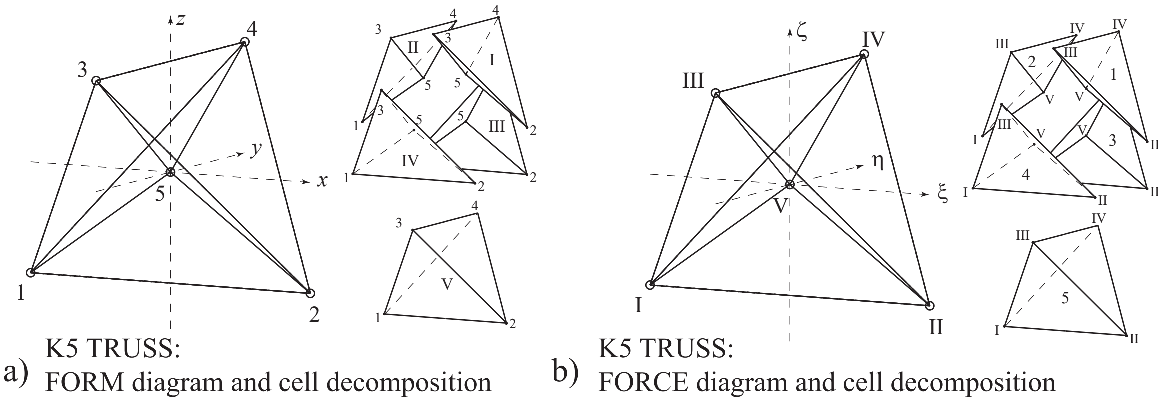
Choosing $H = L^2/2$ leads to a dual K5 identical to the original truss (Fig. 6b), excepting that the dual stress function $\phi = F - H$. Axial forces in bars are then given by $p$ times the area of the faces of the dual object. The outer triangles give forces in the radial spokes, and the inner blade-like triangles give the forces in the outer bars. This much is well-known. The novelty here is that similar structures can now be considered, but with moments.

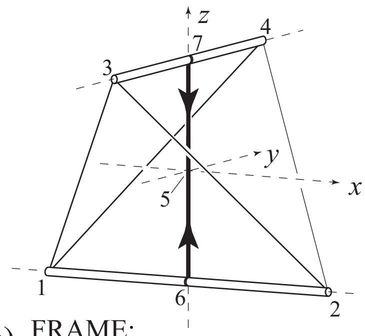
In Fig. 6c, an outer tetrahedron is stressed by a cable between the midpoints of two opposite bars, 12 and 34. Those bars act as beams carrying moments and shears. The structure can be considered to be the edges of a cell complex. One possible cell decomposition is shown in Fig. 6c. The choice here is simply to use exactly the same five cells as previously, but with the central node 5 split into nodes 6 and 7 which are then moved apart towards the outer beams. The four inner cells that originally shared the central node 5 now share bar 657 as a common edge. Many faces are now not plane.

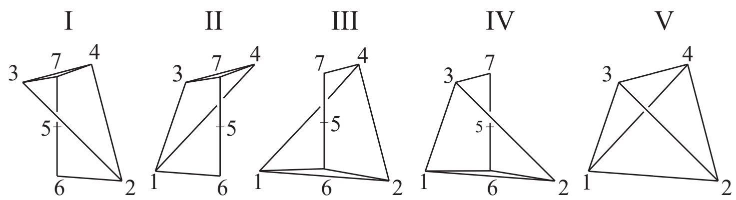
Locally over each cell we may assume a linear stress function. If all joints were fully moment-connected there would exist many possible states of self-stress. Here though we choose to model the case where the four sloping diagonal outer bars (13,23,14,24) that separate the two cross-beams *act purely in axial compression, carrying the same forces as in the original K5 truss*. This can be achieved by using *exactly the same five stress functions as in the truss case*, since the cell neighbourhoods of the sloping outer bars are identical in the two cases. The differences arise only for the central cable and the two opposing cross-beams. In the neighbourhood of these edges, the stress function is discontinuous.

We can set the stress functions to be identical to the earlier case by using $F_5 = H$ again in the Legendre calculations, even though node 5 is no longer a corner. That is, the cell geometries may have changed, but we use the same stress functions as earlier. Since the stress functions are identical to the truss case, the dual nodes are identical to those of the original problem, as shown in Fig. 6b, with nodes I-V dual to the new cells. Fig. 6d shows this dual object projected onto the six orthogonal planes, with areas on planes $\xi\zeta$, $\zeta\eta$ and $\eta\xi$ giving forces, and those on planes $\xi\phi$, $\eta\phi$ and $\zeta\phi$ giving moments. The dual object will also contain cells dual to nodes 6 and 7. These have an interesting geometry, but we do not actually need to consider them here, as the analysis that follows does not depend on the cell decomposition of the dual object.

A section normal to the beam segment 37 intersects cells I, IV, II, V in that order (separated by the faces 2367, 1376, 341, 234). The polygonal loops passing in sequence through the dual nodes I, IV, II, V are shaded in Fig. 6d. The cross-beam thus carries an axial force $pL^2/\sqrt{2}$, a vertical shear of $pL^2$ and has a moment about the origin of $pL^3/2$. Picking the cable tie force to be $W = 2pL^2$, these values all agree with axial, shear and bending moment – at a section on 37 near 7 – of $W/2\sqrt{2}$, $W/2$ and $WL/2$ respectively that may be computed directly by traditional analysis. (The bending moment and axial force create opposing moments about the origin. The bending moment is $WL/2$ and the axial force creates a moment of $WL/4$ about the origin in the opposite direction. The total moment about the origin on this section is thus $WL/4$ as per the polygon $\{\mathrm{I}, \mathrm{IV}, \mathrm{II}, \mathrm{V}\}$ on the $\xi\phi$ plane.




| | $x$ | $y$ | $z$ | $F$ |
|---|---|---|---|---|
| 1 | -L | 0 | -L/√2 | 0 |
| 2 | L | 0 | -L/√2 | 0 |
| 3 | 0 | -L | L/√2 | 0 |
| 4 | 0 | L | L/√2 | 0 |
| 5 | 0 | 0 | 0 | $L^2/2$ |

| Cell | I | II | III | IV | V |
|---|---|---|---|---|---|
| Face | 432 | 341 | 214 | 123 | 234 |
| | 347 | 437 | 126 | 216 | 143 |
| | 2376 | 1476 | 4167 | 3267 | 412 |
| | 4267 | 3167 | 2476 | 1376 | 321 |

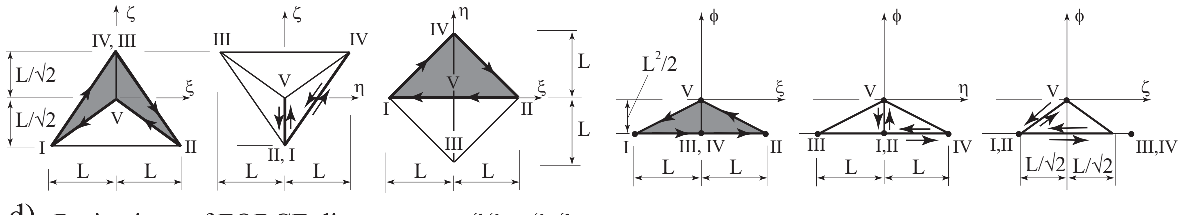



Figure 6: a,b) The simple K5 truss and its dual. c) A tetrahedral frame stressed by a vertical cable carrying tension $W$. Five cells I-V may be defined, many of whose faces are not plane. d) Projections of the dual object (shown in b) onto the six planes in 4D. The shaded polygons are for the stress-resultants in bar 37, corresponding to the loop through dual nodes I, IV, II, V in that order. There is an axial force, a shear force, and a moment about the origin.

All values agree with those readily determined by direct equilibrium computation. Maxwell's mapping, the Legendre transform between body and stress space, thus contains a complete graphic statics capable of representing axial and shear forces with coexistent bending and torsional moments in 3D frames.

## 6. Conclusions

Graphic statics is the embodiment of a Legendre transform between form and force. It is a mapping between the body space and the stress space, as defined by Maxwell's Diagram of Stress [2]. This works in 2D and 3D. The Legendre transform also contains information about moments. The Legendre transform works for 2D/3D continuum stress analysis - as in the example of the Airy beam. It also generalises readily to the discrete case of trusses and frames. Any structural truss or frame may be decomposed into loops. Surfaces - which need not be plane - may be envisioned spanning these loops, and these loops may be envisioned to enclose cells. Piecewise-linear stress functions may be assumed over each cell and these functions may be discontinuous across cell boundaries. The object dual to this cell complex defines an equilibrium set of forces and moments. The Legendre transform thus enables a complete graphic statics for 3D frames, capable of representing all six stress resultants of axial and two shear forces, with torsional and two bending moments.

An earlier paper at IASS [6] showed how to create an equivalent description for frame kinematics. Together, the equilibrium and kinematic descriptions lead to a description of Virtual Work as a "top form", the wedge product of force-moment loops with displacement-rotation loops in a 4D ansatz. The paper here consolidates the equilibrium description.

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