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Free explorations of the polar zonohedron domes

Leila Cristina MENEGHETTI*[,a], Fernando SIMÕES[b], Estevão C. LAURITO[c],
Ruy Marcelo PAULETTI[a], Arthur H. LARA[b]

*,[a] Department of Structural and Geotechnical Engineering, Polytechnic School, University of São Paulo
Av. Prof. Almeida Prado, 83, São Paulo, Brazil
lmeneghetti@usp.br

[b] Faculty of Architecture, University of São Paulo
[c] Hubic - Inovation and Digital Construction, Polytechnic School, University of São Paulo

Abstract

This paper delves into the conceptual and practical applications of Polar Zonohedron Domes (PZD), a term derived from the geometric concept of zonotopes and popularized by artist Rob Bell under the term "Zonotopia". In geometry, zonohedra are defined as centrally symmetric convex polyhedra with rhomboidal faces, which inherently allows the construction of a reticulated PZD using a limited variety of nodes, bars, and faces. This study outlines the mathematical foundations and parametric models critical for the design of PZDs and presents the experimental analysis of two PZD prototypes built using medium-density fiberboard (MDF) and paperboard tubes. The paper details the production of a 6-meter diameter PZD constructed from paperboard tubes and laser-cut plywood connections. The performance of the reticulated dome is also discussed. Results of these investigations enhance our understanding of PZDs' structural capabilities and their potential for broader architectural application.

Keywords: polar zonohedron, domes, gridshells, architectural geometry.

1. Introduction

A polar zonohedron (PZ), a special category of convex polyhedral endowed with central symmetry, formed by the Minkowski sum of equally spaced line segments, resembling a star or umbrella shape. It is symmetric about a line passing through its two poles and about a plane perpendicular to this line through the midpoint. All edges have the same length, and all faces are rhombs. It is usual to align the axis with the vertical direction. The shape of the zonohedron is determined by two parameters: the number of rhombs incident to each pole (n) and the angle (θ) that measures the degree of closure at the poles, ranging between 0 and $\pi/2$. Continuous variation of θ results in a continuum of zonohedral surface, with small θ producing flat shapes and θ near $\pi/2$ resulting in elongated forms (O´Rourke [1]).

The idea of PZ was apparently discovered twice independently. The first time was due to E.S. Federov, in 1885, but his book on mineralogy remained practically unknown for a long time in the West. The idea was reproposed in 1937 by C.H.H. Franklin in the intriguing paper “Hypersolid Concepts and the Completeness of Things and Phenomena”[2].

Early architectural manifestations of PZD can be tracked back to Bruno Taut’s 1914 Glass Pavilion, built in Cologne, Germany. However, despite its acclaim, the pavilion falls short in terms of geometric aesthetics when compared to the elegant surface helices of a PZ. The pavilion's design, resembling a 14-

fold PZ, lacks the mathematical precision that characterizes PZ constructions, resulting in meandering lines that represent a missed opportunity.

Figure 1: Glass Pavilion, Bruno Taut, built in 1914 [3]

Steve Baer, an inventor and designer, delved into zonohedra and Polar Zonohedra (PZ) through mathematical literature, initiating an alternative architecture inspired by these concepts in the 1960s. Baer's pioneering work included the design of buildings with PZ shapes, showcased in his 1970 Zome Primer, featuring a 7-fold PZ. Baer developed a system facilitating the construction of precise physical models, leading to the production of the ZomeToy in 1971. This early iteration evolved into the second-generation Zometool, introduced by Marc Pelletier and Paul Hildebrandt in 1992, sparking broader interest in zonohedra, including PZ. Baer advocated for zonohedra as versatile building components, enabling the creation of diverse structures with a limited inventory of reusable parts. Notably, Baer's central 8-fold PZ dome at the Lama Foundation near Taos, NM, circa 1970, represents one of the earliest architectural-scale PZ implementations. Over subsequent decades, others embraced this approach, leading to a proliferation of PZ homes, greenhouses, sheds, and yurts. The availability of modern computer graphics software further facilitated the visualization of geometric structures, with Russell Towle's software, featured in the Graphics Gallery of the Mathematica Journal in 1996, exemplifying this advancement.

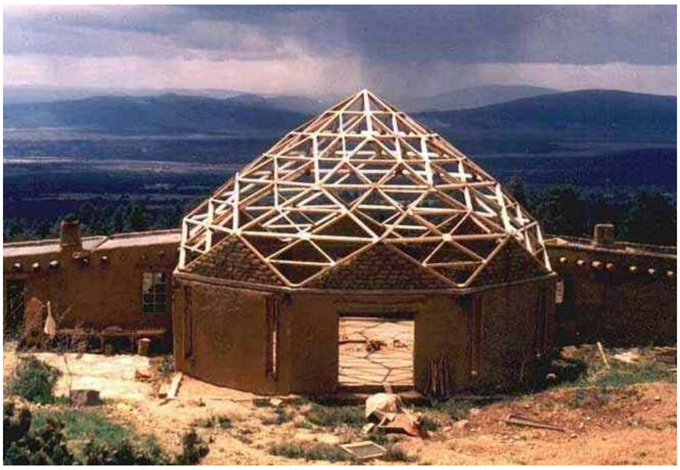

Figure 2: PZ dome at the Lama Foundation, Taos, NM, built circa 1970 [4]

More recently the idea of PZ got a more widespread interest as a structural system. We can recognize the PZ configuration in the lay out of the London's "Gherkin" skyscraper (2003), designed by Norman Foster and Partners, although it has been conceived as a diagrid system. Despite not being a true PZ, the skyscraper's helical structures are meticulously drawn, with panel coloring that accentuates spiral zones.

Figure 3: Gherkin skyscraper, Norman Foster, built in 2003

Polar zonohedra also served as inspiration for artist Rob Bell, who crafted Zonotopia, the realm of Polar Zonohedron Domes (PZD). This concept, derived from the geometrical concepts of zonotopes, influenced Bell's creation of stunning PZD showcased at the Burning Man Festival in the Black Rock Desert, MT. The artist construction, exemplifies a remarkable feat of architectural and artistic ingenuity. Despite facing numerous challenges, including compressed production schedules and logistical obstacles, Bell successfully realized his vision of creating intricate zonohedral structures on the playa. Through innovative construction techniques and meticulous attention to detail, Bell and his team overcame adversity to bring to life the Quasicrystalline Conjunction, a pioneering Spirallohedral pavilion, and the Zomphalo, a sacred altar symbolizing the interconnectivity of organic and geometric elements in nature. Rob passed away in 2019, but his Zomes constitutes a significant milestone in architectural history, demonstrating the fusion of art, science, and engineering in the creation of immersive and transformative experiences.

Figure 4: Zonotopia. Photo by Philippe Glade, cropped from the original (The New Ephemeral Architecture of Burning Man [5])

This article explores polar zonohedra in the construction of domes using parametric modeling to define the dimensions and composition of the pieces. Two domes were produced in a collaboration of the Architecture and Civil Engineering Faculties: a continuous faceted dome, constructed with 3mm medium-density fiberboard (MDF), and lattice structure, comprising recycled paperboard tubes connected by laser-cut plywood joints. The construction of both prototypes enabled an exploration of the architectural possibilities offered by the polar zonohedron. Absence of the continuous faces composing the zones in the lattice zonohedron reduced the stability of the structure, the introduction of additional elements, such as cables interconnecting the connections to stiffen the shape.

2. Materials and method

Two polar zonohedral domes were designed and built as proof of principles. The dome conception, which involves conceptualization, modeling, and the definition of the dimension of the pieces, was carried out using *Rhino 3D* software for parametric modeling via the *Grasshopper* plugin. After modeling, all pieces and connections were produced using laser cutting and a conventional ribbon saw.

2.1. Continuous polar zonohedron dome

The continuous polar zonohedral dome consists of 14 zones and eight trapezoidal segments and one triangular segment each zone (Figure 5). The complete polar zonohedral shape was cut horizontally in the bottom vertex of eighth zone. Thus, the nineth zone pieces became triangular, and at bottom of nineth a belt of 14 rectangular segments was added.

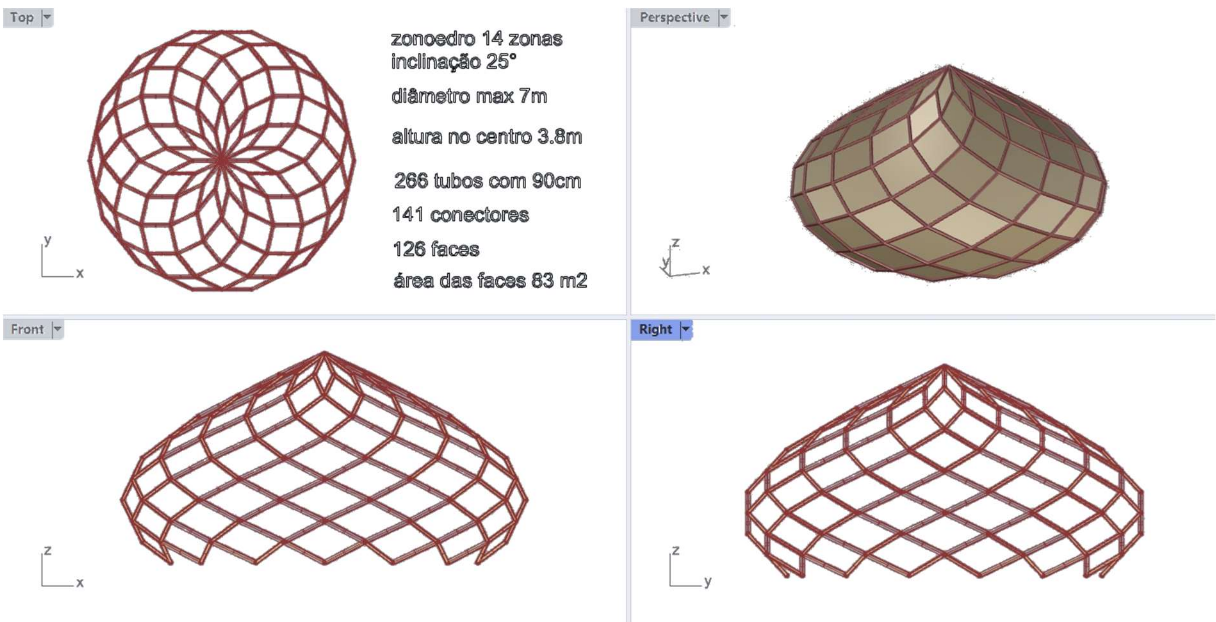

Figure 5: Continuous polar zonohedron design

2.2. Reticulated polar zonohedron dome

The six zones reticulated polar zonohedron dome was built using recycled paperboard tubes, 92 mm diameter and 8 mm thickness. Figure 6 illustrates the designed dome with dimensions in meters. The plywood connection design can be visualized in Figure 7 and 8.

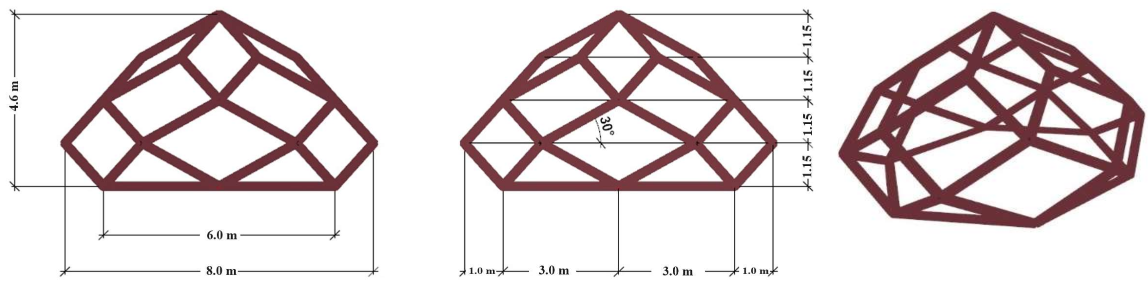

Figure 6: Six zones reticulated polar zonohedron geometry

(a) (b)

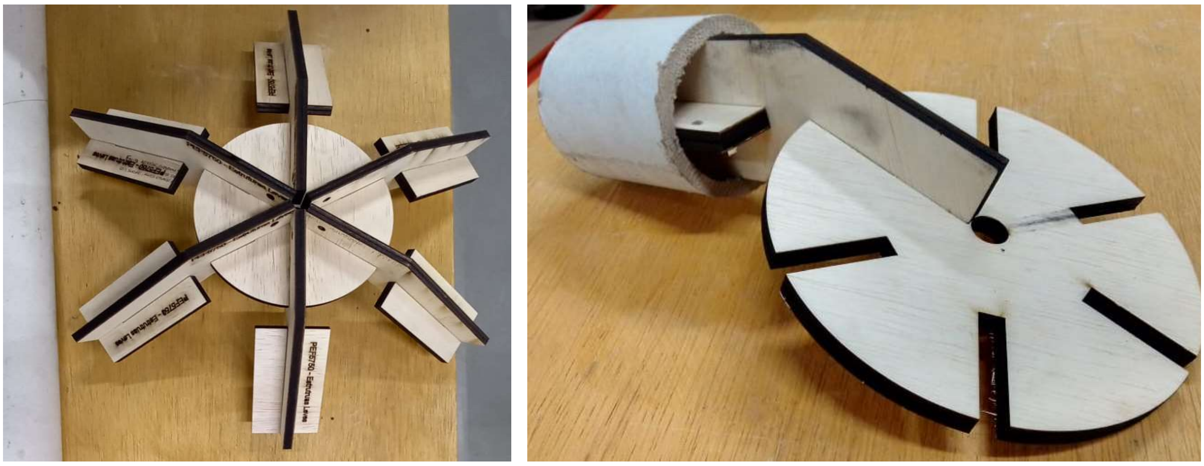

Figure 7: (a) Top connection with 6 slots (one for each zone); (b) Detail of the connection with a prototype tube for assessing clearances and adjustments

(a) (b)

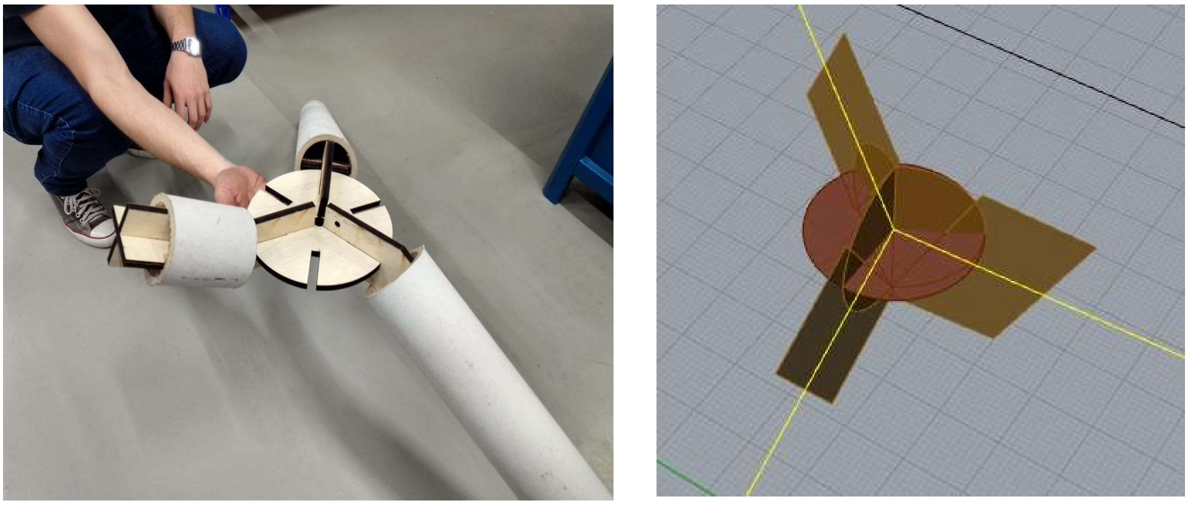

Figure 8: (a) Connection of the intermediate nodes (b) Parametric model of the intermediate connection in *Rhino* 3D – with angles varying according to the number of zones

2.2.1. Mechanics and properties of cardboard tubes

This section presents the mechanical properties experimentally determined associated with a paperboard tubes. Paper, as a material, is generally categorized as orthotropic and exhibits nonlinearity in its viscoelastic behavior. Moreover, it displays an accelerated creep response when exposed to fluctuations in humidity levels. Cardboard tubes, typically crafted from recycled paperboard, introduce an added layer of complexity due to the impact of paper recycling on fiber length and bonding capabilities. When wound spirally, a paper tube transforms into an anisotropic structure, wherein the principal directions of the paper do not align with those of the tube. This inherent anisotropy becomes evident as even external pressure can induce twisting in the tube. Consequently, the design of paper tubes for both conventional and unconventional applications necessitate the utilization of sophisticated methodologies.

The cardboard tubes used in this study have a thickness of 8 mm, an external diameter of 92 mm, and a length of 2200 mm. Test specimens were cut perpendicular to their longitudinal axis with a length of 100 mm, following the recommendations of the ABNT NBR ISO 11093-9:2009 [7]. The tests were conducted using a universal testing machine with a load capacity of 200 kN, where the test specimens were positioned between two parallel smooth plates (one fixed and one movable) and subjected to axial and diametral compression.

Samples subjected to diametral compression, on average, exhibited a maximum load of approximately 850 N, corresponding a displacement of 9.76 mm at the elastic branch of the curve. Beyond this inflection point, the material shows little force increase, but the displacement increases nonlinearly, reaching the ultimate load of 899.49 N at the point of 29.71 mm of maximum displacement, and finally, the test was interrupted. Through visual and tactile inspection of the specimens after the completion of the test, it was possible to note that the deformations produced in the material were responsible for the crushing of the cross-section, causing a distortion from the initial circular shape to an approximately elliptical pattern (Figure 9).

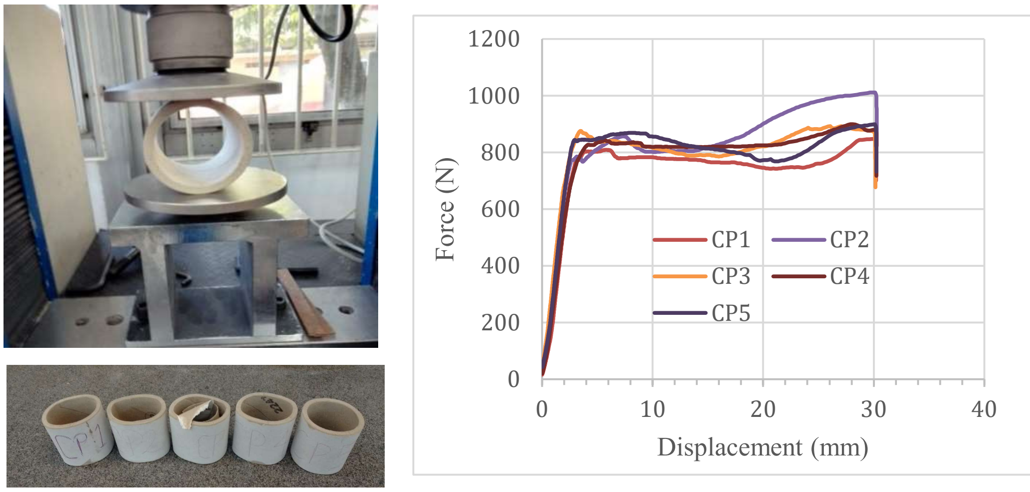

Figure 9: Diametral compression test and specimens after test (left); force x displacement curves (right)

The maximum compressive stress recorded during the tests on cardboard tubes is in the order of 14 MPa, and the maximum specific deformation reached was 0.2 mm/mm. After visual and tactile inspection of the tested specimens, it was observed that they presented edge crushing, in some cases with increased wall thickness, and there were undulations on the external and internal faces, causing deformations with apparent folds (Figure 10).

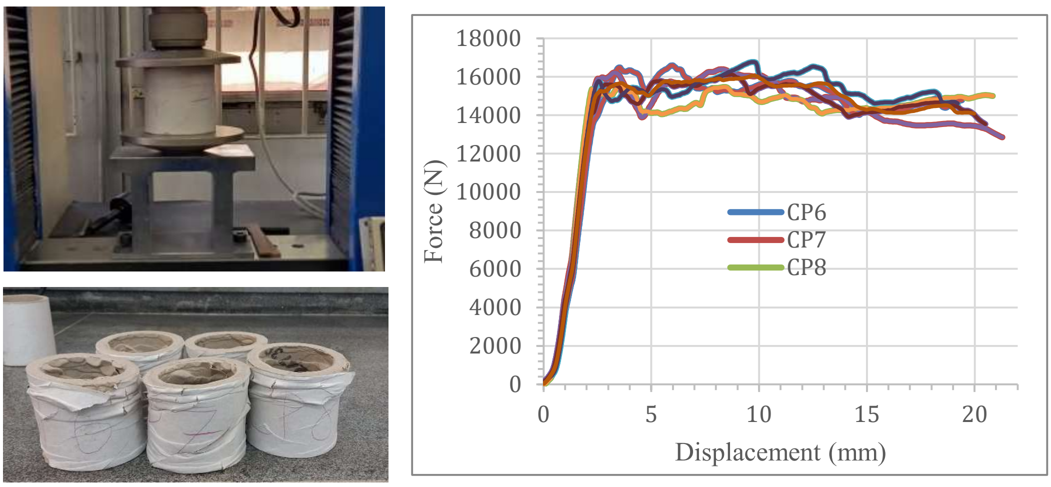

Figure 10: Axial compression test and specimens after test (left); force x displacement curves (right)

Bank and Gerhardt [8] reported the utility of paper board tubes in large structures, some of which have sustained over two decades in outdoor conditions. Often used in temporary exhibitions, these tubes are typically recycled post-use. Additionally, their application in humanitarian disaster relief efforts, notably by Ban for constructing shelters and other structures, underscores their advantages. Paper tubes are favored for their lightweight nature, ease of machining with simple hand tools, straightforward self-assembly, local availability, cost-effectiveness, and minimal environmental impact.

3. Free design explorations and constructions

3.1. Continuous polar zonohedron dome

The continuous polar zonohedron which design was displayed in Figure 5 was assembled using laser cut flat pieces from 3 mm thick MDF boards (Figure 11). Special connectors were included to guarantee continuity between the pieces. A very light and stiff shell was achieved.

Figure 11: Continuous polar zonohedron

Currently, the zonohedron is displayed at the Laboratory of Constructive Culture at the Faculty of Architecture of USP. Its external surface was waterproofed through a painting with castor oil resin (Figure 12).

Figure 12: Continuous polar zonohedron composing the landscape of the Laboratory of Constructive Culture

3.2. Reticulated polar zonohedron dome

A small-scale dome was produced using PVC pipes and MDF connections (Figure 13). The gridshell model is remarkably stiff. A preliminary 1:1 scale prototype with only the upper two layers was also produced as shown in Figure 14.

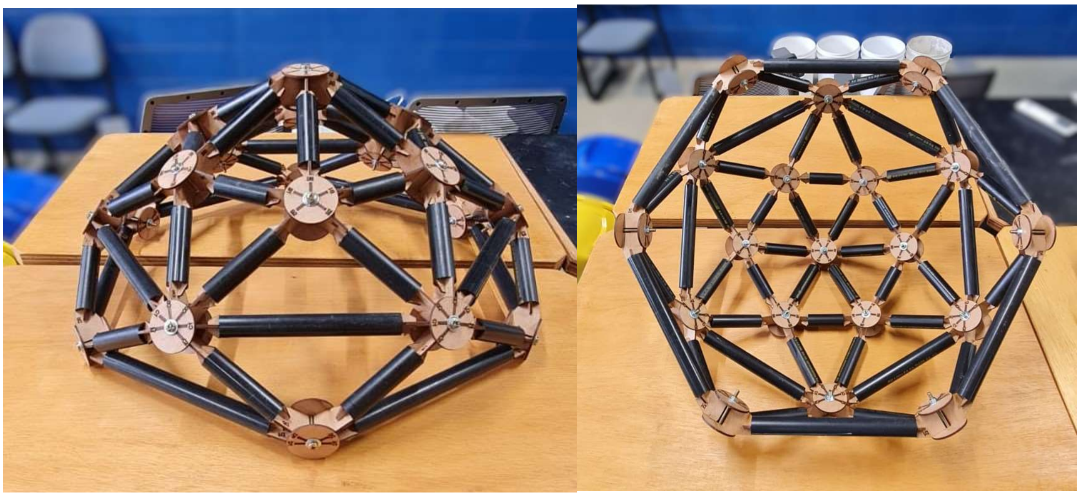

Figure 13: Reticulated small-scalle polar zonohedron

During the assembly of the reticulated zonohedron, it was noticed that the geometry would only achieve stability once all the bars were joined. To overcome the difficulties of instability during construction, cables were added in the circumferential direction passing through the connections. The cables helped stabilize the structure, but there are still significant displacements when the structure is subjected to external forces. In Figure 14, the reticulated zonohedron is seen during assembly.

Currently, a second prototype is been produced embedding the know how so far obtained and results will be present at the IASS Conference.

Figure 14: Reticulated polar zonohedron under construction

4. Conclusion

In conclusion, this paper explores the conceptual and practical applications of Polar Zonohedron Domes (PZDs), The study investigates the underlying mathematical foundations and parametric models crucial for PZD design, alongside the development and experimental analysis of two PZD prototypes using medium-density fiberboard (MDF) and paperboard tubes. While the continuous PZD exemplifies stability and ease of assembly, the reticulated PZD requires additional stabilization methods due to its inherent instability during construction. The results of these explorations contribute to a deeper understanding of PZDs' structural capabilities and their potential for broader architectural applications, setting the stage for further research and development in this innovative field.

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