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Plastic limit load solution for plate under combined tension and bending with any load ratio considering plastic development depth

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Abstract

A plastic limit load solution considering plastic development depth is derived in this paper for plate under combined tension and bending with any load ratio. The results show that for the load ratio within the range of less than -1/6 or greater than 1/6, the stress at the edge of the section increases until the yield strength of the material, and then develops into a full section plastic state (plastic hinge) that includes both the tensile and compressive zones. For the load ratio within the range from -1/6 to 1/6, the stress at the edge of the section will not continue to increase until the yield strength and enter the full section tensile (compression) plastic state. Instead, it will increase to a certain value and gradually decrease to 0, then increase in the opposite direction to the yield strength and eventually develop into plastic hinge state. Given that the non-conservatism for applying the limit load of plastic hinge state to the actual engineering, plastic development depth has been introduced. The new plastic limit load solution considering plastic development depth is suitable for component or structure design and safety evaluation.

Keywords: plastic limit load, plastic development depth, load ratio, combined loading

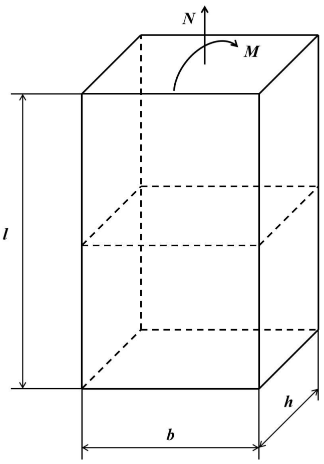
1. Introduction

In plastic analysis, as the load increases, components or structures undergo plastic deformation. When a section reaches the state of a plastic hinge, the corresponding load is termed the plastic limit load. The plastic limit load plays a crucial role in the design, safety assessment, and stability assurance of components or structures. Many textbooks on structural mechanics and plasticity mechanics provide detailed descriptions of calculating the plastic limit load under pure bending loading conditions[1-3]. For loading states where axial force and bending moment act together, the analysis typically focuses on bending-dominant conditions (where the normal stress due to bending in the elastic state exceeds that due to axial force). In such states, as the load increases, the section's stress state transitions from elastic to elastic limit (where the maximum stress in the tensile region reaches the material's yield strength), then enters the plastic state in the tensile region, progressing to the plastic state in the compression region (where the maximum stress in the compression area reaches the yield strength), and finally reaches full section plasticity and the plastic hinge state, at which the load corresponds to the plastic limit load. However, it is still unknown whether sections under axial-dominant conditions (where the normal stress due to bending is less than or equal to that due to axial force) follow the same pattern. Additionally, in actual components or structures, only partial sections are allowed to enter plasticity, necessitating the introduction of the concept of plastic development depth. How the plastic load corresponding to different plastic development depths should be calculated has not yet been reported.

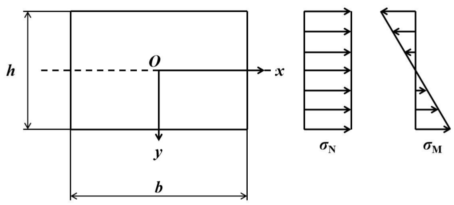
Therefore, this paper conducts a detailed theoretical analysis of the plastic development process for flat plates subjected to combined axial force and bending moment, considering any load ratio. By introducing the concept of plastic development depth, it calculates the plastic load corresponding to different load ratios and development depths, providing parameters for the design and safety assessment of components or structures in engineering.

2. Parameters definition

Figure 1(a) shows a flat plate subjected to an axial force σ_N and a bending moment σ_M . The plate has a length l , a cross-sectional width b , and a cross-sectional height h . The load ratio λ is defined as $\lambda = M/Nh$. Both are subjected to simple loading. Figure 1(b) illustrates the stress distribution of the section under the effects of axial force and bending moment. In the elastic state, the normal stress induced by the axial force is $\sigma_N = N/bh$ and the normal stress induced by the bending moment is $\sigma_M = 6M/bh^2$ .It is stipulated that positive normal stress causes the section to be in tension (positive) and compression (negative). The material is assumed to be an ideal elastic-plastic material with a modulus of elasticity E and a yield strength σ_y .The yield strength in the tensile zone is considered positive, and in the compressive zone, it is negative.



(a) The dimension of flat plate



(b) The stress distribution of section

Figure 1 Parameters definition

3. Analyses of plastic development for plate loaded by positive axial force and moment

3.1. Case 1 for load ratio $\lambda > 1/6$

3.1.1. Elastic period

For $\lambda > 1/6$, the normal stress generated by the axial force σ_N (Figure 2(a)) is less than the normal stress generated by the bending moment σ_M (Figure 2(b)). Therefore, during the elastic stage, the stress state of the section is in a tension-below and compression-above distribution (Figure 2(c)). When the maximum tensile stress in the tension region reaches the material's yield strength, the section is at its elastic limit (Figure 2(d)). The elastic limit axial force and bending moment are as shown in Equation (1):

$$\begin{cases} \sigma_M + \sigma_N = \dfrac{6M}{bh^2} + \dfrac{N}{bh} = \sigma_y \\ \lambda = \dfrac{M}{Nh} \end{cases} \rightarrow \begin{cases} N_e = \dfrac{bh}{6\lambda + 1}\sigma_y \\ M_e = \dfrac{\lambda}{6\lambda + 1} bh^2 \sigma_y \end{cases} \tag{1}$$

3.1.2. Plastic Period for Tension Region

As the load increases, the tension region enters the plastic state (Figure 2(e)). The plastic development depth in the tension region $d = \mu h$, is introduced to calculate the plastic axial force and moment.

Based on similar geometric relationships, the heights h_1 and h_2 can be calculated as shown in Equation (2):

$$\begin{cases} h_1 + h_2 = (1-\mu)h \\ \dfrac{-\sigma_x}{h_1} = \dfrac{\sigma_y}{h_2} \end{cases} \rightarrow \begin{cases} h_1 = \dfrac{-\sigma_x}{\sigma_y - \sigma_x}(1-\mu)h \\ h_2 = \dfrac{\sigma_y}{\sigma_y - \sigma_x}(1-\mu)h \end{cases} \tag{2}$$

The plastic axial force and moment can be determined from the axial force balance equation and the moment balance equation, as shown in Equations (3) and (4):

$$N = \mu bh\sigma_y + \frac{1}{2}b(h_2 - h_1)\times(-\sigma_x + \sigma_y) = \frac{1}{2}bh\left[(1+\mu)\sigma_y + (1-\mu)\sigma_x\right] \tag{3}$$

$$\begin{aligned} M &= \mu bh\sigma_y \times \left(\frac{h}{2} - \frac{\mu h}{2}\right) + \frac{1}{2}bh_2\sigma_y \times \left(\frac{h}{2} - \frac{h_2}{3} - \mu h\right) + \frac{1}{2}bh_1(-\sigma_x)\times\left(\frac{h}{2} - \frac{h_1}{3}\right) \\ &= \frac{1}{12}(1-\mu)(1+2\mu)bh^2(\sigma_y - \sigma_x) \end{aligned} \tag{4}$$

Introducing the load ratio $\lambda = M/Nh$ into the above equations allows expressing the compressive stress σ_x using the tension region's plastic development depth coefficient μ and the load ratio λ , as shown in Equation (5):

$$\sigma_x = \frac{-6\lambda(1+\mu) + (1-\mu)(1+2\mu)}{(1-\mu)(1+2\mu+6\lambda)}\sigma_y \tag{5}$$

Substituting the above into Equations (3) and (4), the plastic axial force and moment expressed by the load ratio λ and the plastic development depth coefficient μ in the tension region are obtained, as shown in Equations (6) and (7):

$$N = \frac{1}{1 + \dfrac{6\lambda}{1+2\mu}} bh\sigma_y \tag{6}$$

$$M = \frac{1}{\dfrac{1}{\lambda} + \dfrac{6}{1+2\mu}} bh^2\sigma_y \tag{7}$$

An analysis of the function's increase or decrease for Equation (5) reveals that for any load ratio greater than 1/6, as the plastic development depth in the tension region increases, the compressive stress gradually increases, and the plastic axial force and moment also increase. When the compressive stress reaches the material's yield strength (Figure 2(f)), the plastic development depth coefficient at this point is as shown in Equation (8):

$$\mu = \frac{1-6\lambda}{4} + \frac{\sqrt{(6\lambda-1)^2 + 8}}{4} \tag{8}$$

Therefore, for any load ratio greater than 1/6, when the plastic development depth in the tension region is less than the limit given by Equation (8), the plastic axial force and moment can be calculated using Equations (6) and (7).

3.1.3. Plastic Period for Compressive Region

As the load further increases, the compressive region also enters the plastic state (Figure 2(g)). The plastic development depth in the compressive region $d_x = \nu h$ is introduced to calculate this phase's plastic axial force and moment.

The plastic axial force and moment can be calculated from the axial force balance equation and the moment balance equation, as shown in Equations (9) and (10):

$$N = \mu bh\sigma_{\mathrm{y}} - \nu bh\sigma_{\mathrm{y}} = bh(\mu - \nu)\sigma_{\mathrm{y}} \tag{9}$$

$$\begin{aligned} M &= \mu bh\sigma_{\mathrm{y}} \times \left(\frac{h}{2} - \frac{\mu h}{2}\right) - \frac{1}{2}b \times \frac{1-\mu-\nu}{2}h \times \sigma_{\mathrm{y}} \times \left(\frac{1}{3} \times \frac{1-\mu-\nu}{2}h + \mu h - \frac{h}{2}\right) \\ &\quad + \nu bh\sigma_{\mathrm{y}} \times \left(\frac{h}{2} - \frac{\nu h}{2}\right) + \frac{1}{2}b \times \frac{1-\mu-\nu}{2}h \times \sigma_{\mathrm{y}} \times \left(\frac{h}{2} - \frac{1}{3} \times \frac{1-\mu-\nu}{2}h - \nu h\right) \\ &= \frac{1}{6}\left(-2\mu^2 - 2\nu^2 + \mu + \nu + 2\mu\nu + 1\right)bh^2\sigma_{\mathrm{y}} \end{aligned} \tag{10}$$

Introducing the load ratio into these equations allows expressing the compressive region's plastic development depth coefficient ν using the tension region's plastic development depth coefficient μ and the load ratio λ, as shown in Equation (11):

$$\nu = \frac{1}{4}\left[(1+2\mu+6\lambda) - \sqrt{f(\mu)}\right] = \frac{1}{4}\left[(1+2\mu+6\lambda) - \sqrt{12\left(4\lambda^2+1\right) - 12\left(\mu - \frac{1-2\lambda}{2}\right)^2}\right] \tag{11}$$

Substituting the above into Equations (9) and (10), the plastic axial force and moment expressed by the load ratio λ and the plastic development depth coefficient in the tension region μ are obtained, as shown in Equations (12) and (13):

$$N = \frac{1}{4}\left[(2\mu - 1 - 6\lambda) + \sqrt{f(\mu)}\right]bh\sigma_{\mathrm{y}} \tag{12}$$

$$M = \frac{1}{4}\lambda\left[(2\mu - 1 - 6\lambda) + \sqrt{f(\mu)}\right]bh^2\sigma_{\mathrm{y}} \tag{13}$$

An analysis of the function's increase or decrease for Equation (11) reveals that for any load ratio greater than 1/6, as the plastic development depth in the tension region increases, the plastic development depth in the compressive region gradually increases, and the plastic axial force and moment also increase. When the sum of the plastic development depths in the compressive and tension regions equals the section height hh, the section's stress distribution exhibits a plastic hinge state (Figure 2(h)), with the plastic development depth coefficient in the tension region as shown in Equation (14):

$$\mu = \frac{1}{2}\left[\sqrt{4\lambda^2+1} - (2\lambda - 1)\right] \tag{14}$$

Substituting Equation (14) into Equations (12) and (13), the plastic limit axial force and moment are obtained, as shown in Equations (13) and (14):

$$N_{\mathrm{p}} = \left(\sqrt{4\lambda^2+1} - 2\lambda\right)bh\sigma_{\mathrm{y}} \tag{15}$$

$$M_{\mathrm{p}} = \lambda\left(\sqrt{4\lambda^2+1}-2\lambda\right)bh^2\sigma_{\mathrm{y}} \tag{16}$$

Thus, for any load ratio greater than 1/6, when the plastic development depth in the tension region is greater than the limit given by Equation (8) but less than that given by Equation (14), the plastic axial force and moment can be calculated using Equations (12) and (13).

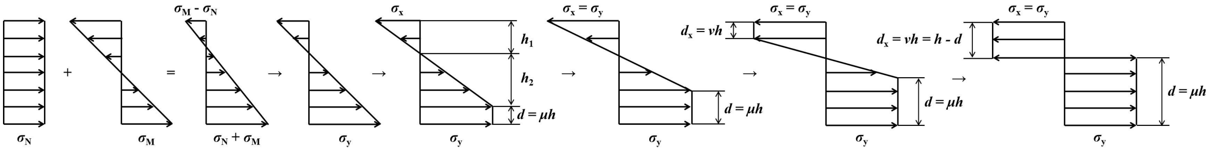


Figure 2 The plastic development analyses for case 1

3.2. Case 2 for load ratio $0 < \lambda < 1/6$

3.2.1. Elastic period

For $0 < \lambda < 1/6$, the normal stress generated by the axial force σ_N (Figure 3(a)) is more than the normal stress generated by the bending moment σ_M (Figure 3(b)). Therefore, during the elastic stage, the stress state of the section has no compressive region, and the tension stress in the lower edge is larger than that in the upper edge (Figure 2(c)). When the maximum tensile stress in the tension region reaches the material's yield strength, the section is at its elastic limit (Figure 3(d)). The elastic limit axial force and bending moment are the same as Equation (1).

3.2.2. Plastic Period for Tension Region

As the load increases, the tension region enters the plastic state (Figure 3(e)). The plastic development depth in the tension region $d = \mu h$, is introduced to calculate the plastic axial force and moment.

The plastic axial force and moment can be determined from the axial force balance equation and the moment balance equation, as shown in Equations (3) and (4). Introducing the load ratio $\lambda = M/Nh$ into the above equations allows expressing the compressive stress σ_x using the tension region's plastic development depth coefficient μ and the load ratio λ, as shown in Equation (5). Substituting the above into Equations (3) and (4), the plastic axial force and moment expressed by the load ratio λ and the plastic development depth coefficient μ in the tension region are obtained, as shown in Equations (6) and (7).

Perform a functional increase/decrease analysis on Equation (5), and it should be noted that the range of load ratio values at this point is $0 < \lambda < 1/6$. Within this range, the variation of tensile stress at the upper edge of the section is significantly different from when the load ratio is greater than 1/6. When the depth coefficient of plastic development in the tensile zone is satisfied with $0 \le \mu \le \frac{1-6\lambda}{4}$, the tensile stress at the upper edge σ_x gradually increases from $\frac{1-6\lambda}{1+6\lambda}\sigma_{\mathrm{y}}$ to $\frac{(1-6\lambda)(9-6\lambda)}{9(2\lambda+1)^2}$ (Figure 3 (f)). When μ is satisfied with $\frac{1-6\lambda}{4} \le \mu \le \frac{\sqrt{1-6\lambda}\left(\sqrt{1-6\lambda}+\sqrt{9-6\lambda}\right)}{4}$, σ_x gradually decrease to 0 (Figure 3 (g)). When μ continues to increase, it becomes negative and changes from tensile stress to compressive stress (Figure 3 (h)) until it reaches the yield strength of the material (Figure 3 (i)), at which point the plastic development depth coefficient μ is the same as Equation (8). However, regardless of the variation of tensile stress at the upper edge of the section, the plastic axial force and bending moment both increase with the increase of plastic development depth in the tensile zone.

Therefore, for any load ratio satisfied with $0 < \lambda < 1/6$, when the plastic development depth in the tension region is less than the limit given by Equation (8), the plastic axial force and moment can be calculated using Equations (6) and (7).

3.2.3. Plastic Period for Compressive Region

The calculation process at this section is consistent with 3.1.3, and the expression is also exactly the same, so it will not be repeated.

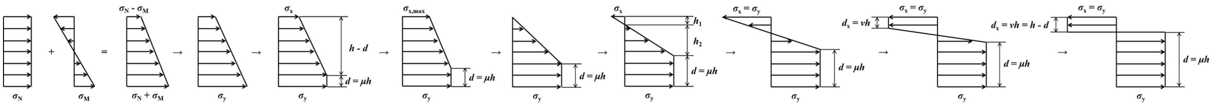


Figure 3 The plastic development analyses for case 2

3.3. Case 3 for load ratio $\lambda = 1/6$

This situation is consistent with the analysis process in 3.2, with the only difference being that during the plastic development process in the tensile zone, the tensile stress at the upper edge of the section changes from 0 to a negative value (compressive stress), and the other expressions are completely consistent with case 2. Here, only a diagram of the plastic development process is provided, as shown in Figure 4.

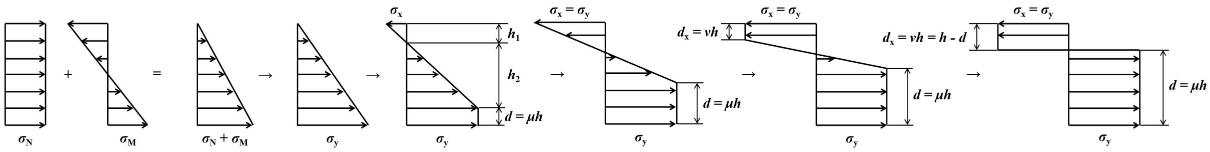


Figure 4 The plastic development analyses for case 3

4. Plastic load expression under any combination of positive or negative axial forces and bending moments

The third section provides a detailed theoretical analysis and formula deduction for a flat plate under the combined action of positive axial force and positive bending moment. For other combinations, the analysis process is similar, and the expression only has a difference of positive and negative signs. Here is a list of the positive and negative values of each parameter under different combinations, as shown in Table 1.

Table 1 Summary of Positive and Negative Parameters

Axial force N	Bending moment M	Yield strength σ_y	Load ratio λ
+	+	+	+
+	-	-	-
-	+	-	-
-	-	+	+

5. Conclusion

This paper conducts a detailed theoretical analysis of the plastic development process for flat plates subjected to combined axial force and bending moment, considering any load ratio. By introducing the concept of plastic development depth, The plastic load corresponding to different load ratios and development depths can be calculated. The results show that for the load ratio within the range of less than -1/6 or greater than 1/6, the stress at the edge of the section increases until the yield strength of the material, and then develops into a full section plastic state (plastic hinge) that includes both the tensile and compressive zones. For the load ratio within the range from -1/6 to 1/6, the stress at the edge of the section will not continue to increase until the yield strength and enter the full section tensile (compression) plastic state. Instead, it will increase to a certain value and gradually decrease to 0, then increase in the opposite direction to the yield strength and eventually develop into plastic hinge state.

Acknowledgements

This template was adapted from the one used for IASS-IACM 2008 with modifications modeled after templates for recent IASS annual symposia.

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