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Form finding of spoke wheel systems with Airy stress polyhedra

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Abstract

This study concerns the application of the dual Airy stress polyhedra to numerical form finding of spoke wheel roof structure. The dual Airy stress polyhedra of form and force diagrams have provided a new perspective on the graphical form-finding of a spoke wheel system in the author's past work. This paper develops the approach further for the nonlinear numeric form finding by generalizing sbp's affine concepts and classifying the form and force diagrams. In theory, the compatible pairs of form and force diagrams can reduce the nonlinear nodal equilibrium equations in terms of coordinates and prestressing forces to those of the dual coordinates of the Airy stress force polyhedron. This paper demonstrates how this perspective can boost form finding process for the spoke wheel structure, and discusses the concept that fully work in the dual coordinates, so that form is more directly converted from Airy stress polyhedron.

Keywords: form finding, spoke wheel roof structures, conceptual design, morphology

1. Introduction

The form finding of a spoke wheel roof structure is a highly nonlinear problem as it requires simultaneously determining the geometry of ring and radial cables, flying struts and compression ring, as well as ensuring equilibrium under prestressing and dead load such that there are only axial forces. The difficulty in solving this form finding problem also comes from the fact that it is usually an underdetermined system. That is, there are, if any, infinite number of solutions in highly nonlinear conditions.

While such difficulty lies in the general form of the problem, the affine method was developed for a non-computational approach and has been applied by Schlaich Bergermann und Partner (sbp) since their early spoke wheel roof structures [1]. The affine concept provides a simple approach to determining a circular or oval spoke wheel structure behaving in self-equilibrium. The idea is described by the affine relationship between tension and compression rings as shown in Figure 3 a). It is a simple but structurally efficient practical solution that yields no bending moment in the outer ring beams in the self-equilibrium state. The sbp first applied the concept to the circular roof for the bullfight arena in Zaragoza. Later, they used it to oval stadium roofs with an analogy to Korbbogen, also known as the three-center arch. Despite such structural advantages and the simple graphical approach not requiring sophisticated nonlinear calculation, the graphic approach using affine concept had a limitation in applicable geometry since the Korbbogen arrangement was the only variant for a non-circular plan. Therefore, it has not been applied to more complex geometry than circular or symmetric ellipse-like shapes. In addition, it had not been thoroughly described how to connect multiple circles sharing a tangent direction with different radii until the author extended the graphical approach by generalizing the affine concept [13]. Other non-computational approach is proposed by Tellier [12] using the funicularity of conics. To determine the forces in a three-dimensional spoke wheel system, the technical reports by sbp describe the general basis of the equilibrium in a spoke wheel system, extracting the vertical plane equilibrium with the relevant horizontal resultants [2]. For more general requirements including undulated rings and other sophisticated design variations, solutions have been resorted to nonlinear form finding methods by Kemmler [3] that can solve the equilibrium and geometric constraints at the same time in probably the simplest manner. Similar concept in which equilibriums and geometric constraints are coupled in terms of unknown forces and coordinates is also developed by Takahashi [4], Martín-Sáiz [5] and Bahr [6]. Computer-aided interactive equilibrium modeling, including the spoke wheel system, has also been explored, for instance, by Lachauer extending the use of the force density method [7]. Vector-based 3D graphic statics was also developed for interactive and real time manipulation of form and forces [10], and further sophisticated by the combnatorial variations for exploring new structural concept [11]. The duality of states of self-stress of planar trusses and the mechanisms of the reciprocal diagram is studied by integrating structural linear algebra and graphic statics by Mitchell and McRobie [8][9].

However, the difficulty or, if not, complexity still remains in the numeric form finding process for spoke wheel structures. The reason is probably because extensive study has not been done on why the problem specific to the spoke wheel system is difficult. The spoke wheel have been used as a challenging example among other types for new form finding methods that are developed for general usage, therefore focus are usually not on the peculiarity of the spoke wheel structure.

The goal of this paper is to provide the a new perspective for the spoke wheel form finding from the particular structure of Airy stress force polyhedron and its 2D projection, that is the force diagram. With the perspective a designer can tailor a suitable set of constraint to effectively reach a desirable solution. The paper also discusses the further evolving concept that fully work in the dual coordinates, instead of the uncertain geometry, so that form is more directly converted from Airy stress polyhedron.

The spoke wheel structures concerned in this paper are limited to the de-facto basic configurations, consisting of either one compression and two tension rings, as shown in Figure 1 or two compression and one tension rings divided by N spoke axes. This paper focuses on the self-stressed equilibrium for the following reasons. The necessary condition for the spoke wheel structure to be stabilized and stiffened by prestressing without assuming a heavy cladding weight is that the configuration should have at least one self-stressed solution. In many cases, the dead weight of the spoke wheel roof is not dominant compared to live load. Therefore, the discussion is not extended to the equilibrium form including dead weight. The structural behaviors or stability issues are also excluded from the scope of this study.

2. Methodology

2.1. General form of form finding equations for a spoke wheel system

For the standard spoke wheel systems that consist of either one compression and two tension rings or two compression and one tension rings divided by N spoke axes, there are $9N$ (= N nodes/ring x 3 rings x 3DOF/node) nodal equilibrium equations for the same number of system DOF and 6N unknown axial forces. Therefore, the general form of form finding of the self-stressed state can be written by

$$F_i(x_j, y_j, z_j, f_k) = 0 \tag{1}$$

where (x_j, y_j, z_j) with $j=1\sim9N$ are nodal coordinate vectors and f_k are the axial forces in the k-th bar element with k=1 to $6N$, while F_i ($i=1\sim9N$) represents the axial force equilibrium equations at the i-th node. Each nodal equilibrium equation is a function of the coordinates of the node and the adjacent nodes and the axial forces in the bars connected to it. Without prescribing any variables, we have $15N$ unknown variables for $9N$ equations. The problem is highly nonlinear because the force components are expressed as products of the force and the directional cosines as functions of coordinates, and it is an underdetermined system. Usually, additional design constraints for geometry and forces are chosen to reach a desirable solution.

$$G_i(x_j, y_j, z_j, f_k) = 0 \tag{2}$$

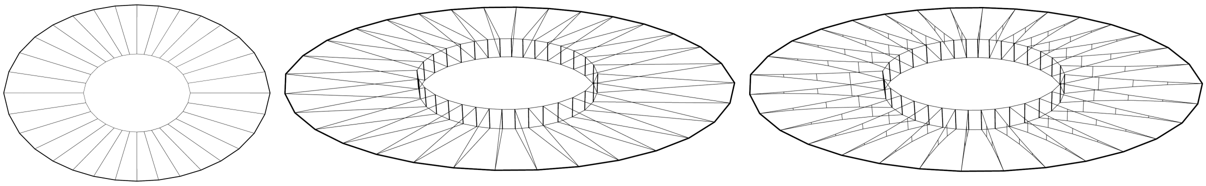
The generalized Newton-Raphson method incorporated with a singular-value-decomposition linear solver can be utilized to solve it if the numbers of the effective equations and the unknown variables are not equal. However, the difficulties lie in the fact that there are so many, possibly infinite, solutions, but they are hard to obtain because of the high non-linearity or the lack of equations. In addition, even if a solution is found, it is often useless because many of the potential solutions are not necessarily desirable from design aspects.

Herein above, the numeric process is described probably the most abstract manner because the contribution of this paper should be valid to any form finding method as far as the nodal equilibrium are the fundamental equations and the unknown design variables are both geometry and forces independently. The independence is important for the following discussion.

The key to solve the system for a usable solution is how to provide wise constraints compatible to the desired design of a spoke wheel. Such constraints include partially prescribed geometry, such as some nodal coordinates or element lengths and/or the axial forces prescribed in particular elements. They constitute additional equations and reduce the gap between the number of equations and variables. In choosing the constraints, however, it is not easy to count for the rank of the system of the linearized equations evaluated in the Newton-Raphson process.

2.2. Three-step approach

The system of equations is high nonlinear but separable into horizontal and vertical equilibriums. Geometric constraints, such as on-the-line and/or on-the-plane controls, cannot always be defined at once. For example, to define a plane for limiting feasible region of a node, the nodes to define the plane must be find first. Therefore, breaking down the problem into three steps as follows reduces the difficulty in numeric convergence as well as help a designer to conceive a feasible configuration. As Figure 1 illustrates, the first step is to find a planar self-equilibrium. The second step is elevate it in to three-dimensional equilibrium including the struts between the rings, either in tension or compression, but excluding the hanger cables between the pairs of spoke cables. In the third step, modify the shape of radial spoke cables by introducing the hanger cables. The solution variables for the second step shall be used for the nonlinear calculation of the third step. Note that the coordinates of the rings should be already known by this step, so that they can be technically removed from the variable unless they are still need to be modified in three dimensional configuration. As such, the phased approach is recommended although it is not always necessary.



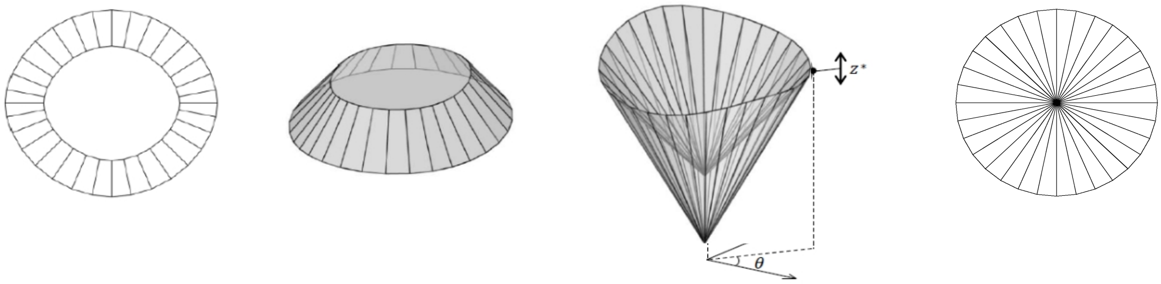
a) b) c)

Figure 1: Three-step approach: a) planar equilibrium, b) three-dimensional equilibrium without hangers, c) three-dimensional equilibrium with hangers

2.3. Max independent variables of planar projection

As discussed in the previous report [15][16], the planar projection of the spoke wheel's self-equilibrium is satisfied if the force diagram represented by two nested cones exists, as shown in Figure 2. A general force diagram shown is not necessarily a regular polygon as in d). All feasible geometry can be generated by converting the Airy stress polyhedron of the force diagram. For a spoke wheel's 2D projection defined by an outer ring, an inner ring and spokes, respectively composed of N elements, the number of geometric variables is $4N$ (= N x 2 rings x 2-dimensional nodal coordinates), and that of the axial forces is $3N$. Because an arbitrarily prescribed geometry does not always have a force set for a self-equilibrium, the apparent system variables for the form finding add up to $7N$. Because the self-equilibrium configuration is not fixed to a location, two DOFs of the system may be prescribed arbitrarily. Because the self-equilibrium configuration is not fixed to a location, and a self-equilibrium force set is scalable, two DOFs of the system and one non-zero force may be prescribed without losing generality. Thus, the number of meaningful unknowns is 7N-3. Since the number of nodal equilibriums in a 2D configuration is $4N$, providing $3N-3$ independent constraints yields the equal number of equations and variables. That is, $4N+(3N-3)=7N-3$.

On the other hand, the number of system variables defining the force polyhedron is $3N+3$ (= (N+1 apex of the cones; the other one is the origin) x 3-dimensional nodal coordinates), and any converted geometry ensures the self-equilibrium. This is a significant reduction of the variables. It indicates that the max number of the independent variables for a planar spoke wheel equilibrium is $3N+3$ under additional design constraints that narrow down the potential design solutions.



a) b) c) d)

Figure 2: Form and force diagrams of 2D-projected spoke wheel structure in self-stressed equilibrium; a) form diagram, b) Airy stress form polyhedron,, c) Airy stress force polyhedron, d) force diagram

2.4. Classification of planar self-equilibrium layouts of spoke wheel systems

The graphic statics and dual Airy stress polyhedra along with the affine concept provide a new perspective to understand the relations between the form and the forces in spoke wheel structures that are essential to direct the form finding to a desirable solution.

2.4.1. The affine class symmetric about two orthogonal axes

The affine concept applied in the spoke wheel roof design by sbp can be appreciated to the greatest extent by interacting with the corresponding force diagrams. Figure 3 shows various two dimensional projections of affine spoke wheel structures that are symmetric about x and y axes on the left, and their force diagrams on the right. All the cases shown here are assumed to have the identical 2D geometry in the upper and lower cable network layers. It is evident from the force diagrams in Figure 3 that the outer compression ring and the inner tension ring are affine in relation to spokes, if and only if the force diagram about the nodes on the outer ring and those on the inner ring are identical and overlaid on each other. Note that the affine relation is not the necessary condition for self-stressed equilibrium of a spoke wheel structure.

a) shows a concentric circular or regular polygonal plan, and b) shows its force diagram that is represented by a regular polygon of the same number of edges with the form diagram and the diagonals connected to the center. This form diagram potentially yields one of the most ideal states of the self-equilibrium in terms of material usage because the all ring force are equal and spoke forces are equal. c) is a non-circular plan, i.e. irregular polygon, but it can be still converted from the regular polygonal force diagram d) which is identical to b). Note that in both forms shown in a) and c), the angles between a spoke cable and two connected ring elements are all equal. In other word, the inner angle of the ring polygon is all equal and the spoke cables are the bisectors of the adjacent ring edges. The selection of geometry is extended as shown in e) if different spoke cable forces are allowed such that the force polygon f) is inscribed in a circle but the polygon edges have different lengths. In the case of e), the spoke cables are still bisectors of the ring polygon's inner angles, but the angles are not equal. Instead of equalizing ring forces, one can equalize the spoke cable forces as shown in the force diagram h). The corresponding form diagram g) does not have bisector spokes in general unless ring forces are equal as well. Lastly, i) is an arbitrary but symmetric non-circular, elliptic, polygonal ring and spokes. Once corresponding force diagram is constructed for one of the rings and the spoke geometry prescribed as shown in j), it is clear that the other ring geometry must follow the identical force diagram. Therefore, the angles between inner ring and spoke cables should be affine to those about outer ring.

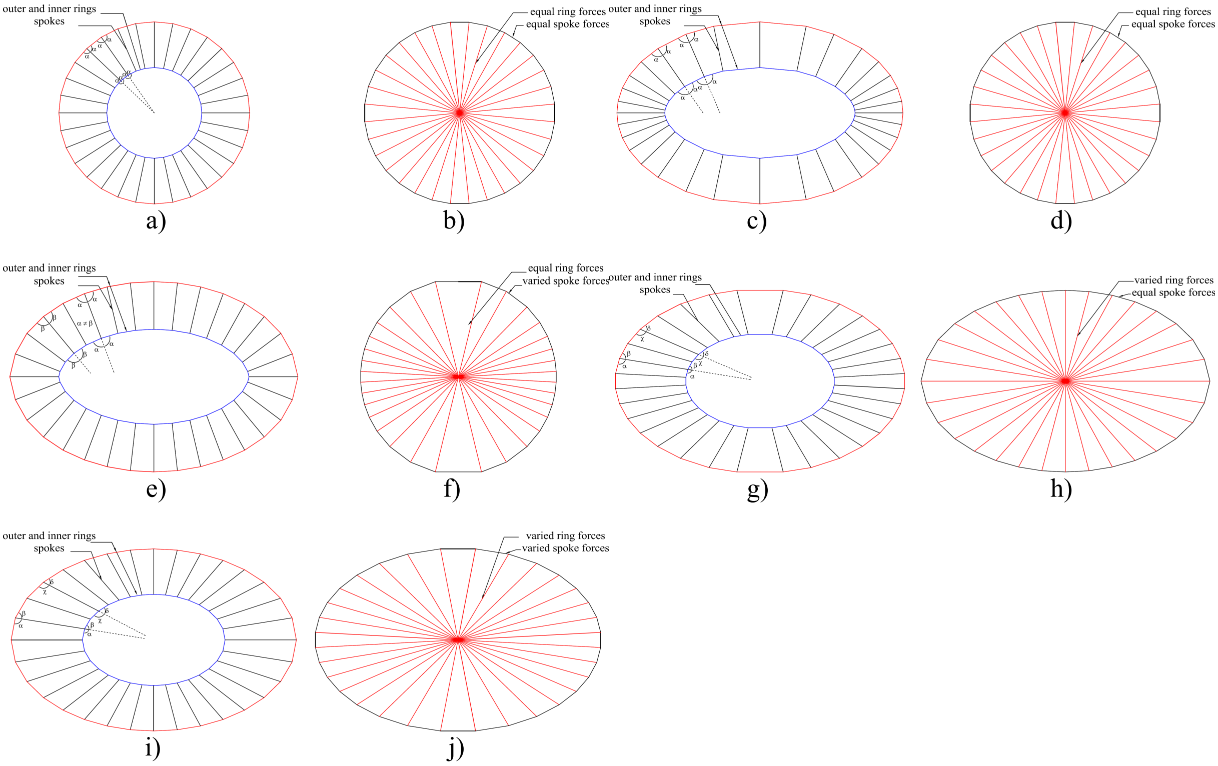


Figure 3: Planer form and force diagrams of affine spoke wheel structure symmetric about two orthogonal axes

While the cases in Figure 3 are symmetric about two orthogonal axes, those in Figure 4 have only one axis of symmetry. Their force diagrams are neither symmetric and in fact they can be created by modifying the similar cases having the two symmetry axes. That is, Figure 4 b) and d) are altered from Figure 3 b) and f) by translating the center point of the ring forces, and Figure 4 f) is altered from Figure 3 f) by transforming the polygonal edges.

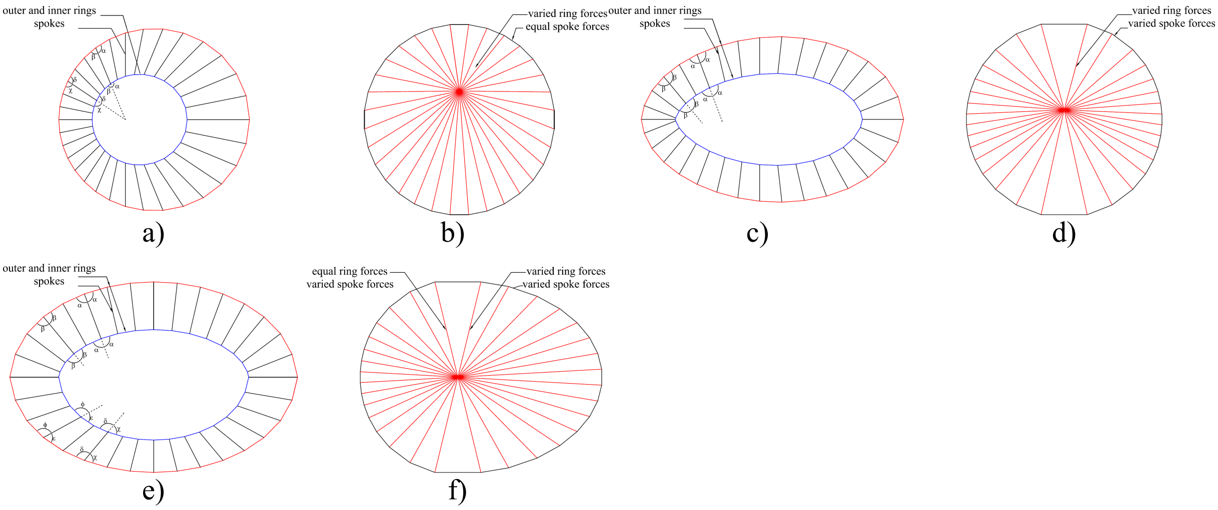


Figure 4: Planer form and force diagrams of affine spoke wheel structure symmetric about one axes

Figure 5 a) shows a completely arbitrary and irregular ring geometry, but it can be in self-stressed as far as the corresponding force diagram can be developed as shown in Figure 5 b). It may be a counter-intuitive, but it is possible that the ring with an irregular polygon still be able to self-stressed by ring equal forces as shown in Figure 5 c) and d). It can be be also approximated by the rings compatible with equal ring forces as well as spoke forces as shown in Figure 5 e) and f).

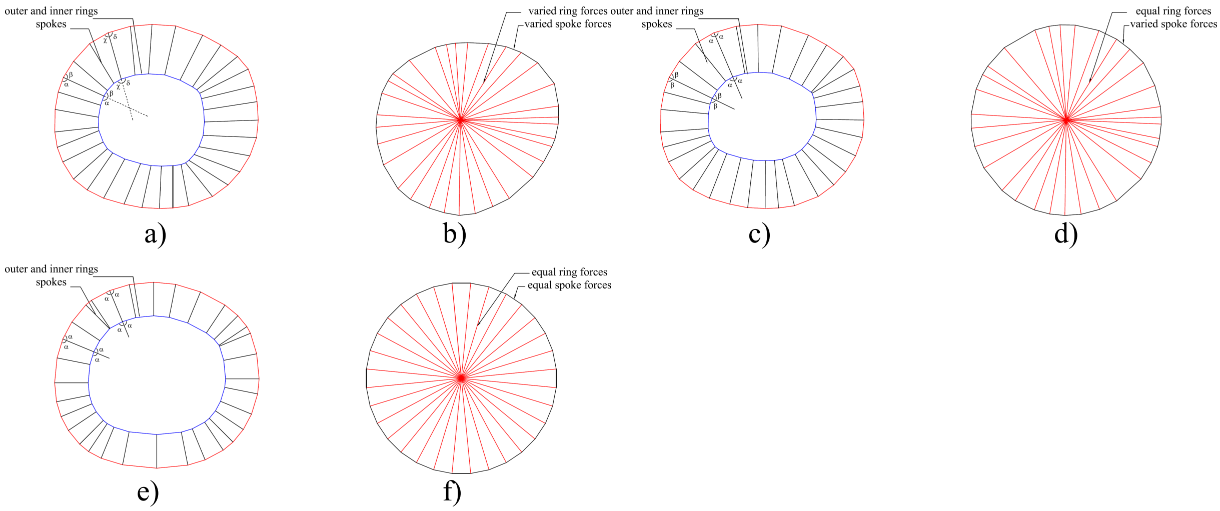


Figure 5: Planer form and force diagrams of affine spoke wheel structure without symmetry

2.4.2. The non-affine class

Figure 6 shows planar self-equilibrium layout where the inner and outer ring geometry are not affine. Such non-affine equilibrium can be developed by sliding the centers of outer and inner ring forces away from each other. Figure 6 b) is altered from Figure 3 f), while Figure 6 d) is altered from Figure 5 b).

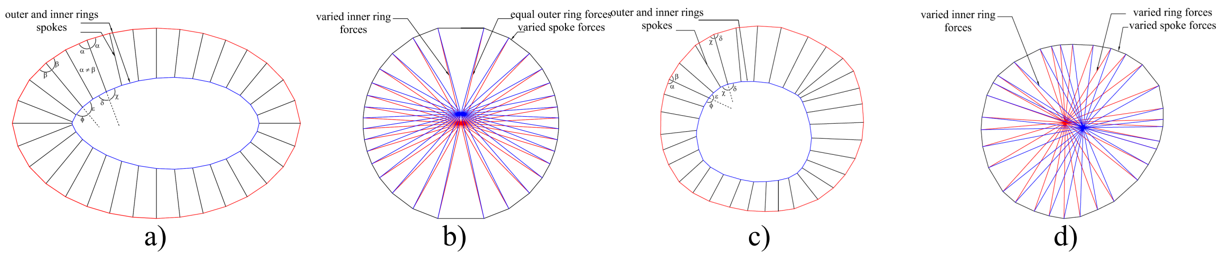


Figure 6: Non-affine spoke wheel planer layouts

2.5. Constraints compatible with classes of spoke wheel systems

As demonstrated in the previous subsection, the system, the prescribed topology satisfying the nodal equilibriums, has infinite variations of solutions. However, these solutions are bounded within the invariant structure of the force diagram, i.e., the topology of its Airy stress force polyhedron described in Section 2.3. Therefore, in practice, there may be only a few essential approach patterns that a designer should know when form-finding a spoke wheel structure. That is, how to tailor the constraints for your desired shape.

The considerations a designer would make in the course of form-finding for a spoke wheel structures, besides the system type and topology, include but are not limited to the listed below:

1. What is the design priority: shape or structural efficiency?
2. Is the plan symmetric or not? Or even irregular?
3. Are the shapes of the outer compression ring(s) and inner tension ring(s) similar?
4. How strict is the requirement for the plan shape?
5. Is your ring undulated?
6. ….

The corresponding conceptual design approach would be as follows;

A1. If the structural efficiency matters the most, consider applying the force diagram of a regular polygon shape as in Figure 3 b), d), and Figure 5 f), or at least that with equal diagonals as in Figure 3 f) and Figure 5 d). Recall that these force diagrams do not limit the form to be symmetric but must be an affine class like Figure 5 f), for example.

A2. When symmetry about two orthogonal axes is required, the affine class is the only choice. However, the non-affine class works if symmetry is needed only about one axis. In this case, the center of the radial force lines must move perpendicularly to the symmetry axis. It is important to note that an irregular configuration can be developed using either the affine or non-affine class, but applying the non-affine class would increase the irregularity.

A3. If exact similarity is required, two rings must be concentric, but the forces will not be equal. Choosing the bisected affine class is crucial to ensure equal ring forces. In this case, the ring shapes differ, although not so far.

A4. If the ring geometry is strictly predetermined, the remaining freedoms are to choose either the direction of the spikes or the compression forces in the outer ring. Theoretically, one can prescribe both variables mixedly as long as they are independent. But practically speaking, they should be appropriately prescribed, understanding that two adjacent ring element forces uniquely determine the direction of the spoke in between. One cannot arbitrarily specify the inner ring shape because only one force polygon represents the spoke forces shared by two sets of radial force lines for outer and inner rings, as depicted by Figure 6 b) and d) in red and blue. In other words, there should be no gap between the two Airy stress force polyhedral cones. The choice for the internal ring is only either affine or not. If affine, the only independent variable remaining is the z^* coordinate of the apex of the polyhedral cone, i.e., the size of the hole because (x^*, y^*) coordinates of the cone are equal to those of the outer ring. If not affine, there are three variables (x^*, y^*, z^*) coordinates of the polyhedral cone's apex, which determines the projection point where all the radial force lines of the inner rings connect.

A5. This paper does not extend the discussion to the details of the rings' undulation, although the next section demonstrates a case.

As explained in Section 2.3, the maximum number of independent constraints a designer can define for a basic spoke wheel discussed in this paper is 3N-3. If fewer constraints are provided, the system of the nonlinear equations is underdetermined and overdetermined if more constraints are provided.

3. Applications

3.1. General case

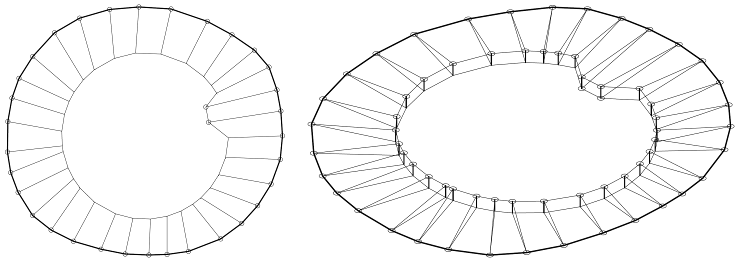
As one of the most general cases, the 3D form finding of irregular configuration introduced in Figure 5 a) is demonstrated step by step. Note that the constraints tailored in this demonstration are one example of many possible combinations.

3.1.1. Step 1/3: finding 2D self-stressed equilibrium

Assume the outer ring geometry and spoke cable arrangement are given as strict requirements. The inner ring geometry cannot be specified arbitrarily; the directions of the spoke cables are predefined except for the one to enable equilibrium at the loop end. Inner ring geometry is prescribed with minimum constraints: position (hole size) and directions of one element, which is parallel to the corresponding outer ring element. The force in this inner ring is prescribed to be equal in magnitude to the corresponding compression ring element. Refer to Figure 7 a). The planar equilibrium is found in Figure 5 a). Initially there are 4N in-plane equilibriums for 4N+3N unknowns. Among 4N rings' nodal coordinates, 2N for compression ring are prescribed and the directions are constrained for N-3 tension ring elements, and 2 node are specified, counting additional 4 (=2x2) prescribed DOF. One of the compression ring forces and one of the spoke cable forces are prescribed, and these add up to 3N+3 (= N-3+4+2) prescriptions and constraints, effectively reducing the variables down to 4N-3. This gives a unique self-stressed solution.

3.1.2. Step 2/3: Finding 3D self-stressed equilibrium without hangers

Convert the determined geometry coordinate as known, and specify, for example, the ratio of inner tension ring force distribution and lengths of the struts so that the vertical positions of the struts are determined. Alternatively, the top and bottom z-coordinates of the strut positions can be specified so that the in-plane spoke forces spread to upper and lower spoke cables to ensure equilibrium. Figure 8 a) shows the 3D self-equilibrium without hangers.



a) b)

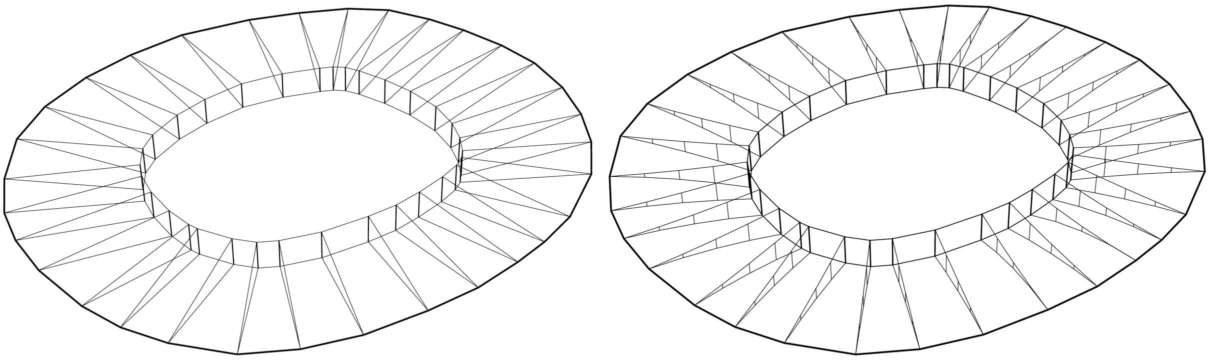
Figure 7: Input data for form finding steps of a spoke wheel with arbitrary ring shape

3.1.3. Step alternative to 1/3+2/3: direct 3D input

Figure 7 b) shows an alternative input data for directly finding a 3D equilibrium, skipping finding a planar equilibrium. The constraints described in 3.1.1 and 3.1.2 should be combined together.

3.1.4. Step 3/3 3D self-stressed equilibrium with hangers

Introduce hangers connecting upper and lower spoke cables. After the planer equilibrium geometry is established, geometric constraints for the hangers are easy to apply. A quick thought gives the radial spoke cables funicular shapes as follows. Let h denote the number of hangers connecting a pair of upper and lower radial spoke cables in one axis plane. Assuming that the forces are constant along each hanger as prescribed, consider the upper and lower half of the system separately. The half system has h+1 modified spoke cable forces and h modified z-coordinates of the joints under hx2 nodal equilibrium equations. Specifying one more variable, for instance, the slope for one segment, yields a unique solution. Figure 8 b) shows the determined self-equilibrium form with hangers.



a) b)

Figure 8: 3D spoke wheel with arbitrary ring shape in self-equilibrium

3.1.5. Step additional to undulate ring

As shown in in Figure 9, undulation can be applied. Four vertical supports are added to undulate the roof.

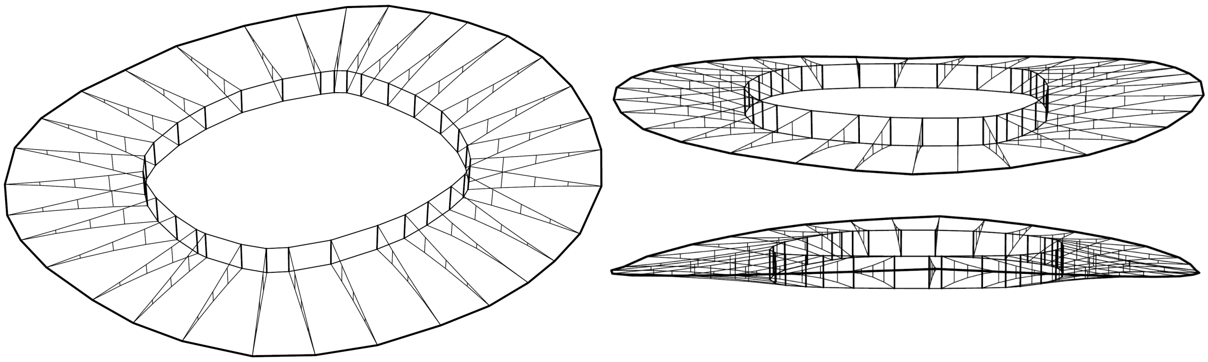


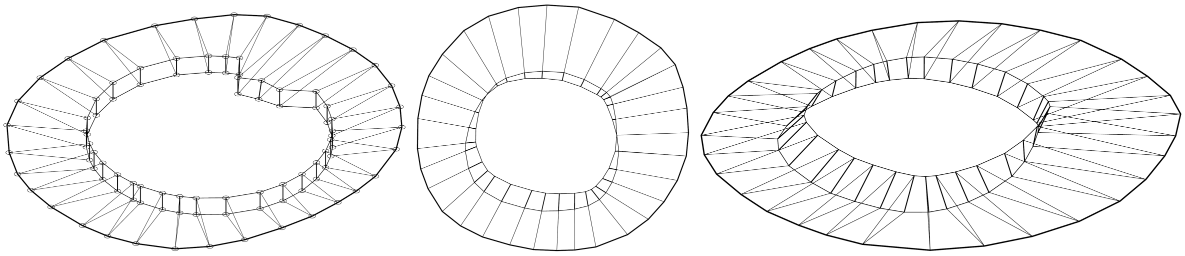
Figure 9: 3D undulated spoke wheel with arbitrary ring shape in self-equilibrium

3.2. Symmetric layout

The three-dimensional self-stressed equilibrium of a symmetric case having various ring and spoke forces determined through the same process as above is presented in Figure 1.

3.3. Composition of an affine and a non affine inner rings in 3D configuration

A composition of different tension ring shapes are demonstrated. The input data defining different classes of the equilibrium geometry is shown in Figure 10 a). For the upper tension ring, one element is set parallel to the outer ring beam and its force is specified to make it affine. For the lower one, the directions of the two elements are specified so that one non-affine configuration is generated. Figure 10 shows the b) top and c) perspective views of the 3D self-equilibrium form determined without hangers.



a) b) c)

Figure 10: Tension rings of different class composed in one 3D spoke wheel with arbitrary ring shape

4. Discussions

This paper has derived practical guidelines to tailor the general framework of the form-finding method for spoke wheel structures. They are based on an in-depth understanding of the characteristic shape of Airy stress form polyhedron and the relation between the form and force diagrams of the self-stressed spoke wheel structures. Recalling that planar self-equilibrium can be constructed using graphic statics, as shown in Section 2.4, one may question why the nonlinear calculation is necessary. Graphic statics proved a powerful tool for developing planar self-stressed spoke wheel geometry. On the other hand, the way a designer prepares the constraints is only sometimes intuitive and completely graphical in the proposed method. However, when solving three-dimensional equilibriums, the nonlinear numeric procedure has some advantages over the conventional construction using graphic statics. The numeric procedure can generate a 3D self-equilibrium model directly, and the selection of variable prescriptions does not change the procedure. In addition, the numeric procedure can be integrated with mathematical minimizing techniques. Techniques to minimize errors can provide an approximated solution to the design intent if the designer's initial ideas are not entirely compatible with the spoke wheel's self-stressed configuration. The author already demonstrated the basic concept in the past works for the regular polygon force diagram [13][15]. For the general case, it is yet to be developed. Although it will not be as simple as the regular polygon force diagram, expressing the form by the force polyhedron is potentially applicable to any part of geometric constraints.

5. Conclusion and future works

This paper has demonstrated that a general numeric form-finding method can leverage the form and force diagrams obtained by graphic statics and Airy stress polyhedra to quickly converge to a desirable solution for a self-stressed spoke wheel structure. The above-discussed alternative method directly applying the Airy stress polyhedra will be conceived in future work. The geometry is expressed by the coordinates of the force polyhedra so that self-equilibrium is satisfied without solving the nodal equilibriums. The challenge in future work will be to establish a flexible framework for any shape of the force diagram that differs from the basic configuration.

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